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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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By virtue of the properties of $\lambda(\psi)$ mentioned above, Σ always exists as a closed convex curve, and reduces to a point-circle, or a straight line of finite length in the case in which $\lambda(\psi)$ is negative in the interior of an angle of extent π. The curvature at a point on Σ and the origin are on the same side or opposite sides of the enveloping tangent at the point according as the corresponing value of $\lambda(\psi)$ is positive or negative. The tangents at those points of Σ which correspond to such values of ψ as are roots of the equation $\lambda(\psi) = 0$, pass through the origin. It follows, therefore, that Σ is a curve to which the origin is exterior, and the tangents to which from the origin include an angle whose magnitude is less than π.*

Now let us measure θ from θ_1 to $\theta_1 + 2\pi$, where θ_1 is $-\pi$ or zero according as the real positive axis intersects Σ or not. Let the tangents from the origin to Σ be $\theta = \alpha$, and $\theta = \beta$, where $0 \leqslant \beta - \alpha < \pi$.

Denote by C the closed contour formed by

$$|u| = a, \ |u| = b, \ \theta = \beta + \varepsilon, \ \theta = \alpha - \varepsilon, \qquad \ldots \quad (2.5)$$

where ε is a positive number so chosen that

$$\alpha - \varepsilon > \theta_1, \ \beta + \varepsilon < \theta_1 + 2\pi, \text{ and } \beta + 2\varepsilon - \alpha < \pi,$$

and a and b are so chosen that Σ lies entirely within the contour C.

Now defining u^z as $e^{z \log u}$ where that value of the logarithm is chosen whose imaginary part lies between θ_1 and $\theta_1 + 2\pi$, we observe that the function $g(u)\, u^z$ is analytic on C, so that

$$c(z) = \frac{1}{2\pi i} \int_C g(u)\, u^z \, du$$

is an integral function of z $\qquad \ldots \quad (2.6)$

It is evident that the contour of integration may be deformed into the circle C_1, whose equation is $|u| = k + \delta, \quad \delta > 0,$

---

* This inequality is essentially due to the fact that the supplement of the angle for which $\lambda(\psi)$ is negative is always less than π, and is in fact less than or equal to $\pi - \delta$, where δ is some positive number.

so that

$$c(n) = \frac{1}{2\pi i}\int_{C_1}\left(\sum_{0}^{\infty}\frac{c_n}{u^{n+1}}\right)u^n\,du, \qquad (n = 0, 1, 2, 3, \ldots\ldots),$$

$$= c_n \qquad \ldots \quad (2.7)$$

and since $\phi(z) \not\equiv 0$, $c(z)$ is not identically zero.

Also

$$|\,c(z)\,| \leqslant \frac{1}{2\pi}\int_{C_1} |\,g(u)\,u^z\,du\,|,$$

so that

$$c(z) = O(e^{L\rho}),$$

where L is some constant. ... (2.8)

It has, therefore, been shown that $c(z)$ satisfies the conditions (1.5) and (1.6). Also by virtue of (1.5), $H(\psi)$ is a continuous function of ψ, which satisfies the inequality

$$H(\psi) + H(\psi + \pi) \geqslant 0. \qquad \ldots \quad (2.9)$$

Now it remains to prove that $H(\pi/2) + H(-\pi/2) < \pi$.

To do this we observe that

$$|\,c(z)\,| \leqslant M\left[\int_{C} |\,u^z du\,|\right], \text{ where } M = \max.\left|\frac{g(u)}{2\pi i}\right| \text{ on } C, \qquad \ldots \quad (2.10)$$

$$\leqslant M\left[\left|\int_{\alpha-\varepsilon}^{\beta+\varepsilon} a e^{\rho(\log a\cos\psi - \theta\sin\psi)}\,d\theta\right| + \left|\int_{\alpha-\varepsilon}^{\beta+\varepsilon} b e^{\rho(\log b\cos\psi - \theta\sin\psi)}\,d\theta\right| + \left|\int_{a}^{b} e^{\rho(\cos\psi\log R - (\alpha-\varepsilon)\sin\psi)}\,dR\right| + \left|\int_{a}^{b} e^{\rho(\cos\psi\log R - (\beta+\varepsilon)\sin\psi)}\,dR\right|\right].$$

Therefore

$$c(\rho e^{i\pi/2}) = O(e^{-\rho(\alpha-\varepsilon)}),$$

and

$$c(\rho e^{-i\pi/2}) = O(e^{\rho(\beta+\varepsilon)}).$$

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Hence

$$H\left(\frac{\pi}{2}\right) \leqslant -\alpha + \varepsilon, \text{ and} \qquad \dots \quad (2.11)$$

and

$$H\left(-\frac{\pi}{2}\right) \leqslant \beta + \varepsilon, \qquad \dots \quad (2.12)$$

so that

$$0 \leqslant H\left(\frac{\pi}{2}\right) + H\left(-\frac{\pi}{2}\right) \leqslant \beta + 2\varepsilon - \alpha < \pi. \qquad \dots \quad (2.13)$$

By virtue of a well-known theorem of Carlson, the function $c(z)$ is unique. This completes the proof of the direct theorem.

3. Now we shall prove the converse proposition. That is, we shall suppose that there exists an integral function $c(z)$ satisfying (1.5), (1.6), and (1.7), and then show that

$$\phi(z) = \sum_0^\infty \frac{c_n z^n}{n!},$$

is an integral function of order 1 and type positive, and that the minimum value of $\lambda(\psi)$, as defined by (1.2), is negative.

Let $c(z) = \sum_0^\infty \frac{a_n z^n}{n!}$. Then the series $\sum_0^\infty \frac{a_n}{\xi^{n+1}}$ is absolutely and uniformly convergent in the region lying exterior to the circle D, $|\xi| = l + \delta$, where $\delta > 0$, and l is the maximum value of $H(\psi)$ in the range $|\psi| \leqslant \pi$.

Therefore

$$f(\xi) = \sum_0^\infty \frac{a_n}{\xi^{n+1}} = \int_0^{\infty(\psi)} c(z)\, e^{-\xi z}\, dz \qquad \dots \quad (3.1)$$

is an analytic function of $\xi\,(= x_1 + iy_1)$ in the region lying exterior to the envelope Γ of the family of lines

$$x_1 \cos\psi - y_1 \sin\psi = H(\psi), \quad (|\psi| \leqslant \pi). \qquad \dots \quad (3.2)$$

By virtue of (2.13), Γ exists as a closed convex curve and the perpendicular distance between its tangents which are parallel to x_1-axis is less than π.

Now consider the Dirichlet's series

$$F(s) = \sum_{0}^{\infty} c(n)\, e^{-sn} = \sum_{n=0}^{\infty} e^{-sn} \sum_{\nu=0}^{\infty} \frac{a_\nu n^\nu}{\nu!}, \quad \ldots \quad (3.1)$$

$$= \sum_{n=0}^{\infty} e^{-sn} \sum_{\nu=0}^{\infty} a_\nu \frac{1}{2\pi i} \int_D \frac{e^{n\xi}\, d\xi}{\xi^{\nu+1}},$$

$$= \sum_{n=0}^{\infty} e^{-sn} \frac{1}{2\pi i} \int_D e^{n\xi} f(\xi)\, d\xi = \frac{1}{2\pi i} \int_D f(\xi) \frac{d\xi}{1-e^{-(s-\xi)}}$$

$$= \frac{1}{2\pi i} \int_{\Gamma'} f(\xi) \frac{d\xi}{1-e^{-(s-\xi)}},$$

each inversion of order of summation and integration being easily justified, and Γ' being the curve parallel to Γ and outside it at an arbitrarily small distance ε, so that the tangents to Γ' which are parallel to the real axis are at a distance less than π apart.

The singularities of the function $\dfrac{1}{1-e^{-(s-\xi)}}$ are poles at the points $s = 2k_1\pi i + \xi$, where k_1 is an integer or zero. Now with respect to each point of the set $s = 2k_1\pi i$, describe a curve Γ_1, identical with Γ', and situated with respect to this point in the same way as Γ' is situated with respect to the origin in the ξ-plane. The region lying exterior to all the curves Γ_1's so drawn is a simply-connected one, and the function $F(s)$ is an analytic function of s throughout this region.

It is evident that $F(s) = c_0$, when $\sigma \to \infty$. In order to study the behaviour of $F(s)$ as $\sigma \to -\infty$, suppose $\sigma < -(l + \delta)$, and

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then

$$F(s) = \frac{-1}{2\pi i}\int_D f(\xi)\,\frac{e^{s-\xi}}{1-e^{(s-\xi)}}\,d\xi$$

$$= \frac{-1}{2\pi i}\int_D f(\xi)\left(\sum_{n=1}^{\infty} e^{n(s-\xi)}\right) d\xi$$

$$= -\sum_{n=1}^{\infty}\frac{e^{ns}}{2\pi i}\int_D e^{-n\xi}\left(\sum_{\nu=0}^{\infty}\frac{a_\nu}{\xi^{\nu+1}}\right) d\xi$$

$$= -\sum_{n=1}^{\infty} e^{ns}\sum_{\nu=0}^{\infty}\frac{(-n)^\nu a_\nu}{\nu!} = -\sum_{n=1}^{\infty} c(-n)\,e^{ns},$$

so that $F(s) = 0$, when $\sigma \to -\infty$.

Now F (s) being a periodic function with a period of $2\pi i$, it is sufficient for our purpose to study this function in a strip of breadth 2π drawn parallel to the real axis. Let us choose the strip $t = \alpha_1$, $t = \alpha_1 + 2\pi$, where $\alpha_1 = -\pi$, or zero, according as one of the curves Γ_1's is intersected by the line $t = 0$, or not. It is clear that this strip has one, and only one, of the curves Γ_1's lying entirely within it, and that the boundaries of the strip do not intersect any member of this system of curves.

Now effecting the transformation

$$x = e^s, \qquad (x = re^{i\theta}, \text{ and } \alpha_1 \leqslant \theta \leqslant \alpha_1 + 2\pi)$$

we find that

$$J(x) = \frac{1}{x}\sum_{n=0}^{\infty}\frac{c(n)}{x^n} = \sum_{0}^{\infty}\frac{c(n)}{x^{n+1}} \qquad \ldots \quad (3.5)$$

is an analytic function of x in the region lying exterior to the curve A which corresponds to Γ_1 in the strip $\alpha_1 \leqslant t \leqslant \alpha_1 + 2\pi$. Since $F(s) = c_0$, when $\sigma \to \infty$, J (x) vanishes at infinity, and since $F(s) = 0$, when $\sigma \to -\infty$, $J(x)$ is regular at the origin.

Therefore

$$\phi(z) = \sum_{0}^{\infty} \frac{c(n) z^n}{n!} = \frac{1}{2\pi i} \int_{A_1} J(x) e^{zx} dx, \qquad \dots \quad (3.6)$$

where A_1 is any closed contour which includes A within it.

Suppose $t = \beta_1$, and $t = \beta_2$, where $\alpha_1 < \beta_1 < \beta_2 < (\alpha_1 + 2\pi)$ and $\beta_2 - \beta_1 < \pi$, are the tangents to Γ_1 in the strip $\alpha_1 \leqslant t \leqslant \alpha_1 + 2\pi$. Then the vectors $\theta = \beta_1$ and $\theta = \beta_2$ are the tangents to A in the x-plane.

Let A_1 be replaced by the following contour, say, A_2.

$$|x| = a_1, \ |x| = b_1, \ \theta = \beta_1 - \varepsilon_1, \ \theta = \beta_2 + \varepsilon_1, \qquad (\varepsilon_1 > 0),$$

where a_1, b_1, and ε_1 are so chosen that A lies entirely within A_2, and

$$\beta_2 + 2\varepsilon_1 - \beta_1 < \pi.$$

Then

$$|\phi(\rho e^{i\psi})| \leqslant M_1 \int_{A_2} |e^{zx} dx|, \text{ where } M_1 = \max. \left|\frac{J(x)}{2\pi i}\right| \text{ on } A_2,$$

$$= M_1 \int_{A_2} e^{r\rho(\cos(\theta+\psi))} |dx| = O(e^{\rho K}), \qquad \dots \quad (3.7)$$

where K is some constant.

As the variation in θ over the contour A_2 is less than π, we can always find one value of ψ, say ψ_1, such that on A_2 $\cos(\theta + \psi_1)$ remains less than $-\delta_1$ where δ_1 is some positive number. It follows, therefore, that

$$\phi(\rho e^{i\psi_1}) = O(e^{-\delta_2 \rho}), \qquad \dots \quad (3.8)$$

where δ_2 is some positive number.

Since $\phi(z)$ is not identically zero, is of order $(e^{K\rho})$, and satisfies the relation (3.8), $\lambda(\psi)$. as defined by (1.2), is continuous, and its minimum value is negative, and consequently its maximum value is positive, Hence $\phi(z)$ is an integral function of order 1 and type positive, and satisfies the condition (1.4).

This proves the converse proposition.

4. Let us now make a few observations.

(i) *In the notation of § 2, $H(\psi)$ possesses the following properties* :—

$$H(0) = \log k, \qquad \ldots \quad (4.1)$$

$$H(\pi) = -\log k', \qquad \ldots \quad (4.2)$$

where k' is the radius of convergence of Taylor's series for $g(u)$ valid in the neighbourhood of the origin ;

$$H\left(\frac{\pi}{2}\right) = -\alpha; \qquad \ldots \quad (4.3)$$

$$H\left(-\frac{\pi}{2}\right) = \beta. \qquad \ldots \quad (4.4)$$

Since k is the type of the function $\phi(z)$, $\overline{\lim} \mid c(n) \mid^{1/n} = k$. But by virtue of a known result,*

$$\overline{\lim} \mid c(n) \mid^{1/n} = \overline{\lim} \mid c(\rho) \mid^{1/\rho} = e^{H(0)}.$$

Hence

$$H(0) = \log k.$$

From (3.4) we have $g(u) = -\sum_{n=1}^{\infty} c(-n)\, u^n$, so that

$$\frac{1}{k'} = \overline{\lim} \mid c(-n) \mid^{1/n} = \overline{\lim} \mid c(-\rho) \mid^{1/\rho} = e^{H(\psi)},$$

and hence we have the relation (4 2). Also since $k \geqslant k' > 0$,

we have
$$H(0) + H(\pi) = \log \frac{k}{k'} \geqslant 0.$$

To prove (4.3) and (4.4), we observe that, since ε is arbitrary, (2.11) and (2.12) imply the following inequalities

$$H\left(\frac{\pi}{2}\right) \leqslant -\alpha, \; H\left(-\frac{\pi}{2}\right) \leqslant \beta.$$

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* G. Pólya : 'Untersuchungen uber Lucken und Singularitaten von Potenzreihen,' *Mathematische Zeitschrift* Bd. 29, p 606 I have proved a more general theorem than Pólya's. I propose to publish this shortly.

Suppose, if possible, that

$$H\left(\frac{\pi}{2}\right) = -\alpha - \delta_1, \text{ and } H\left(-\frac{\pi}{2}\right) = \beta - \delta_2,$$

where δ_1 and δ_2 are positive numbers. Then following the method of § 3, we can show that $g(u)$ is an analytic function of u, in the region lying exterior to a closed convex curve Σ, to which the origin is exterior, and the tangents to which from the origin are $\theta = \alpha + \delta_1$, and $\theta = \beta - \delta_2$. But this is impossible, since it is known* that the vectors $\theta = \alpha$, and $\theta = \beta$ contain at least one singularity of $g(u)$ each. This establishes the relations (4.3), and (4.4).

(ii) It is evident by now that, *under certain circumstances*, given $\lambda(\psi)$, it should be possible to know $H(\psi)$ for all values of ψ, and *vice versa*. Here is an example.

A necessary and sufficient condition that $\lambda(\psi)$, *as defined by* (1.2) *be* $(K + R\cos\psi)$, *where* $K < R$, *is that there should exist an integral function* $c(z)$ *which assumes the values* $c_0, c_1, c_2, \ldots$ *for* $z = 0, 1, 2, \ldots$, *and which satisfies the following asymptotic equality* :—

$$\overline{\lim_{\rho\to\infty}} \frac{\log |c(\rho_o^{i\psi})|}{\rho} = \sin\psi\left(\sin^{-1}\left(\frac{K}{R}\sin\psi\right)\right) -$$

$$\cos\psi\left\{\frac{-K\cos\psi + \sqrt{R^2 - K^2\sin^2\psi}}{R^2 - K^2}\right\}, \quad (|\psi| \leqslant \pi).$$

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* Pólya : *loc. cit.*, p. 585.

ON THE DIVERGENCE OF THE UNIT NORMAL.

BY R. VAIDYANATHASWAMY.

If a surface V^2 is immersed in a Euclidean space E^3 of three dimensions, then we know from Dupin's Theorem, that at any point of V^2, the sum of the curvatures of any two perpendicular normal sections is a constant equal to the sum of the principal curvatures. This is in fact, an immediate consequence of Euler's formula

$$\frac{1}{\rho} = \frac{\cos^2 \alpha}{\rho_1} + \frac{\sin^2 \alpha}{\rho_2},$$

for the curvature of a normal section making an angle α with a principal section. Following Weatherburn, we may call the sum of the principal curvatures, *the first curvature* of the V^2 at the point considered. While the Gaussian (or *second*) curvature, which is equal to the product of the principal curvatures, is an *intrinsic* invariant of the manifold V^2, the first curvature is not so, as it has reference to immersion in a Euclidean space of higher dimensions.

The congruence of normals to V^2 is also normal to a one-parameter family of surfaces, namely, the surfaces *parallel* to V^2. This congruence determines a unit vector-field in E^3 which we may call *the unit normal.* Weatherburn has proved* that the divergence of the unit normal in E^3, evaluated at any point of V^2, is equal to *minus* the first curvature of V^2 at the point.

The object of the paper is to prove that a corresponding result can be stated for any Riemannian manifold V^n, of n dimensions, immersed in in a second Riemannian manifold V^{n+1}, of $n + 1$ dimensions, it being understood, of course, that the metric in V^n is that which is induced by the metric of V^{n+1}. It is known that equations analogous to Euler's hold for this case, with a suitable interpretation of the *normal curvature* of a curve lying in V^n, immersed in a V^{n+1}.† This suggests by analogy, the concept of "first curvature of a V^n immersed in a V^{n+1}." Again, if at each point of the V^n, we draw the unique geodesic of V^{n+1} which is normal to the V^n at the point, we obtain the *normal geodesic congruence* of V^n. If we lay out a constant length from V^n along these normal geodesics, the locus of the points so obtained is a second V^n, which is

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* Weatherburn's *Differential Geometry*, Chapter XII, page 226.

† Eisenhart : *Riemannian Geometry*, page 154.

parallel (or *geodesically parallel*) to the first, and has the same normal geodesic congruence.* This congruence determines a unit vector-field in V^{n+1}, which we call as before, *the unit normal*. It will be shown that the divergence of this unit normal in V^{n+1}, is equal, at any point of V^n, to the negative of its first curvature.

I. *The Co-ordinate System.*

In Eisenhart's *Riemannian Geometry*, the geometry of a V^n immersed in a V^{n+1} is developed without assuming any special relations between the co ordinates in V^n and those in V^{n+1}, and an expression is found for the co-variant derivate of the unit normal. It would doubtless be possible to prove our result from this expression, but the proof would be rather complicated. I shall use instead a simpler and more natural system of co-ordinates. Let $x^1, x^2, \ldots, x^n$ be co-ordinates in V^n chosen in any manner. Let the positive direction of the normal geodesic at any point of V^n be fixed arbitrarily, so that the positive direction of the normal at any other point will be determined by continuity. Let P be any point $(x^1, x^2, \ldots, x^n)$ in V^n, and P′ a point on the normal geodesic at P, so that the geodesic distance PP′ (reckoned positively or negatively, according as PP′ is or is not the positive direction of the normal at P) is x^0. Then we may take $x^0, x^1, \ldots, x^n$ as co-ordinates of p' in V^{n+1}. If g_{ij} $(i, j, = 1, 2, \ldots, n)$ is the metrical tensor in V^n, and G_{ij} $(i, j, = 0, 1, \ldots n)$ in V^{n+1}, we evidently have, in this system of co-ordinates:

$$G_{oo} = 1; \quad G_{ok} = 0; \qquad (k = 1, 2, \ldots, n).$$

Also it is clear that g_{ij} results on putting $x^0 = 0$ in G_{ij}. Also, let $\gamma_{i,jk}$, $\Gamma_{i,jk}$ be the Christoffel symbols of the first kind in V^n, V^{n+1} respectively, and γ^i_{jk}, Γ^i_{jk}, those of the second kind. It is clear in the first place that γ^i_{jk} $(i, j, k = 1, 2, \ldots n)$ is the result of putting $x^0 = 0$ in Γ^i_{jk}. We have further:

$$(1) \quad \Gamma^o_{jk} = G^{o\lambda}\,\Gamma_{\lambda,\,jk} = \Gamma_{o,\,jk} = -\tfrac{1}{2}\left(\frac{dG_{jk}}{dx^0}\right)$$

$$(2) \quad \Gamma^i_{jo} = G^{i\lambda}\,\Gamma_{\lambda\ jo} = \tfrac{1}{2}\,G^{i\lambda}\left(\frac{dG_{j\lambda}}{dx^0}\right);$$

since $G^{io} = 0$ $(i \neq 0)$, the summation here may be limited to the values $1, 2, \ldots, n$ of the doubled index λ.

* For proof, see page 57, *Ibid*.

We may note that in the above co-ordinate system x^0 = constant represents the family of V^n's geodesically parallel to the given V^n, the equation of which is $x^0 = 0$. Also, it is clear that the unit normal-field in V^{n+1} is the gradient of the scalar field x^0; or, more accurately, since the gradient is the co-variant vector $(x^0),_i$, it is the vector $(x^0)^i = G^{i\lambda}(x^0),_\lambda$, obtained by raising the index in $x^0),_i$. It is clear that $(x^0)^0 = 1$, $(x^0)^i = 0$ $(i = 1, 2, \ldots, n)$.

II. *The Normal curvature and the second fundamental form of V^n.*

Given a curve C in V^n, we may take its equations in the form:

$$x^i = x^i(s) \qquad (i = 1, 2, \ldots, n),$$

where s is the length of the arc measured from a fixed point on the curve. Then $\lambda^i = \frac{dx^i}{ds}$ are the components of the unit tangent vector of the curve. The co-variant derivate $\lambda^i,_j$ does not exist as λ^i is not defined except on the curve, but the co-variant arc-derivate:

$$\mu^i = \lambda^i,_j \lambda^j = \frac{d^2 x^i}{ds^2} + \gamma^i_{jk} \lambda^j \lambda^k,$$

exists, and is a vector perpendicular to λ^i, and equal to its arc-rate of change. μ^i is thus analogous to the principal normal, so that the curvature of the curve would be given by:

$$\frac{1}{\rho} = \sqrt{\mu^i \mu_i} = \sqrt{g_{ij}\mu^i \mu^j}.$$

We may call μ^i *the vector curvature* of the curve.

If now we consider C as a curve in V^{n+1}, it will be represented by the same equations together with the additional equation $x^0 = 0$. The tangential unit vector will have the same components as before, and in addition, the normal component $\lambda^0 = 0$. The vector curvature will have the same components as before, and in addition, the normal component

$$\mu^0 = \frac{d^2 x^0}{ds^2} + \Gamma^0_{jk} \lambda^j \lambda^k \qquad (j, k = 0, 1, \ldots, n)$$

$$= -\tfrac{1}{2}\left(\frac{dG_{jk}}{dx^0}\right)_0 \lambda^j \lambda^k,$$

where the suffix 0 indicates that x^0 is to be put equal to zero after differentiation. Thus the vector curvature of C in V^{n+1} is composed of its vector curvature in V^n, and a normal component μ^0, which, unlike the former, depends solely on the tangential unit vector λ^i, and not on its arc-rate of change. We call μ^0 *the normal curvature* of C in V^n.

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Writing

$$b_{jk} = -\tfrac{1}{2}\left(\frac{d\,G_{jk}}{dx^0}\right)_0,$$

it follows that the normal curvature at a point in the direction determined by a vector v^i, is given by

$$\frac{1}{\rho} = \frac{b_{ij}v^i v^j}{g_{ij}v^i v^j}.$$

We call b_{ij}, by analogy with the usual differential geometry, *the second fundamental form* of V^n. From its definition, we see that it is $-\frac{1}{2}$ times the variation of the metrical tensor, when we pass from V^n to a consecutive geodesically parallel manifold.

The stationary values of the normal curvature (or *the principal curvatures*) at a point are roots of the determinant equation

$$\left|\frac{1}{\rho}\ g_{ij} - b_{ij}\right| = 0.$$

If the roots $\frac{1}{\rho_1}, \frac{1}{\rho_2}, \ldots \frac{1}{\rho_n}$, of this equation are all distinct, they determine n principal directions at the point, defined by n unit vectors $\lambda^i_{(k)}$ $(k = 1, 2, \ldots, n)$. From the usual theory of two quadratic forms, we know that these unit vectors are mutually perpendicular, and also satisfy the relations

$$b_{ij}\,\lambda^i_{(p)}\,\lambda^j_{(q)} = 0 \quad (p \neq q).$$

From these equations, it follows immediately that the normal curvature in the direction of the unit vector $v^i = \sum_k \cos\alpha_k\,\lambda^i_{(k)}$, which makes angles α_k with the principal directions, is given by

$$\frac{1}{\rho} = \frac{\cos^2\alpha_1}{\rho_1} + \ldots + \frac{\cos^2\alpha_n}{\rho_n}.$$

This is the analogue of Euler's equation. By writing down Euler's equations for n mutually perpendicular unit vectors, and adding, we immediately see that

the sum of the normal curvatures of n mutually perpendicular directions at any point of V^n is constant and equal to the sum of the principal curvatures at the point.

Accordingly, we may call the sum of the principal curvatures, the *first curvature* of the V^n; since the principal curvatures are roots of the equation

$$\left| \frac{1}{\rho} g_{ij} - b_{ij} \right| = 0,$$

the value of the first curvature is

$$\frac{b_{ij} \times (\text{minor of } g_{ij})}{| g_{ij} |} = b_{ij}\, g^{ij}$$

$$= -\tfrac{1}{2}\, g^{ij} \left(\frac{dG_{ij}}{dx^0}\right)_0.$$

III. It remains to calculate the divergence of the unit normal, and to show that it leads to the same expression.

The unit normal N^i, being the gradient of the scalar x^0, has the components

$$N^0 = 1, \qquad N^i = 0 \quad (i = 1, 2, \dots, n).$$

Hence its co-variant derivate in V^{n+1} is

$$N^i_{,j} = \frac{d N^i}{dx^j} + \Gamma^i_{jk} N^k$$

$$= \Gamma^i_{j0}$$

$$= \tfrac{1}{2} G^{i\lambda} \left(\frac{d\, G_{j\lambda}}{dx^0}\right) \quad \text{(from 1)}.$$

Thus the divergence of the unit normal is

$$N^i_{,i} = \tfrac{1}{2}\, G^{ij} \left(\frac{dG_{ij}}{dx^0}\right) \qquad (i, j = 0, 1, \dots, n)$$

$$= \tfrac{1}{2}\, G^{ij} \left(\frac{dG_{ij}}{dx^0}\right) \qquad (i, j = 1, 2, \dots, n)$$

If calculated at a point of V^n, this becomes

$$\tfrac{1}{2}\, g^{ij} \left(\frac{dG_{ij}}{dx^0}\right)_0$$

and this differs only in sign from the expression obtained for the first curvature. Thus, it is established that the first curvature is the negative divergence of the unit normal.

SOME ARITHMETICAL IDENTITIES.

By A. A. Krishnaswami Ayyangar, M.A., L.T.

1. Let $[x]$ denote the greatest integer in x. It is proposed to study in this note certain identities* involving expressions of the form

$$\left[\frac{an+b}{p}\right] \text{ and } \left[(an+b)^{1/p}\right]$$

where a, b, p, n are all positive integers.

2. Let $$u_n = \left[\frac{an+b}{p}\right] \text{ and } v_{n+1} = u_{n+1} - u_n;$$

when a and b are both less than p, it is easily seen that v_{n+1} can take only one of two values 0 or 1 according as

$$\left.\begin{array}{r} a(n+1)+b \equiv a,\ a+1, \ldots (p-1) \ldots (\text{mod } p) \\ \text{or} \\ 0, 1, 2, \ldots (a-1) \ldots (\text{mod } p) \end{array}\right\} \text{(A)}$$

For, let $an + b = kp + k_1$ (k, k_1 being integers and $k_1 < p$), then $a(n+1) + b = kp + k_1 + a$, where $(k_1 + a) < 2p$.

Hence,

$$\left[\frac{a(n+1)+b}{p}\right] - \left[\frac{an+b}{p}\right] = \left[\frac{k_1+a}{p}\right] < 2$$

i.e. $$\left[\frac{k_1+a}{p}\right] = 0 \text{ or } 1 \text{ according as}$$

$$a \leqslant (k_1 + a) < p \text{ or } p \leqslant k_1 + a < p + a.$$

Now, the p-congruences in (A) are not all soluble for n, if a, p have a common factor. If h is the highest common factor of a and p, the

* The identities of this note were suggested to the present writer by similar identities proposed by the late S. Ramanujan in Q. **723** (*J. I. M. S.* Vol. VII, p. **240**). *Vide* also *Collected papers* : p. **332**. This question has been solved by the late H Balakram in pp. 357—358. Vol X., *J. I. M. S.*, but the general principle underlying such identities is not brought out there.

soluble congruences selected from (A) are given by

$$\left.\begin{array}{l}\frac{a}{h}\cdot n \equiv -\left[\frac{b}{h}\right], -\left[\frac{b}{h}\right]+1, \ldots \\ -\left[\frac{b}{h}\right]-\frac{a}{h}, -\left[\frac{b}{h}\right]-\frac{a}{h}+1, \ldots\end{array}\right\}\left(\bmod \frac{p}{h}\right) \qquad \ldots \text{(B)}$$

where the upper line contains $\frac{p-a}{h}$ and the lower line $\frac{a}{h}$ consecutive integers.

The values of n (mod p/h) which satisfy (B) are some permutations of the integers 0, 1, 2, ... $(p/h - 1)$; and for $\frac{a}{h}$ of these values of n, $v_{n+1} = 1$, while for the rest, $v_{n+1} = 0$. Thus the values of 'v' form a period of p/h elements a/h of which are unity and the rest zero. We shall call this period, the period associated with $\left[\frac{an+b}{p}\right]$.

Example. To find the period associated with $\left[\frac{9n+11}{15}\right]$.

Here $a = 9$, $p = 15$, $h = 3$, $b = 11$, and the congruences to be solved are

$$\left.\begin{array}{c} 3n \equiv -3, -2 \\ \text{or} \\ -6, -5, -4 \end{array}\right\} \pmod 5$$

Hence, $v_{n+1} = 0$ or 1 according as $n \equiv 4, 1$ (mod 5,

or $\qquad n \equiv +3, 0, +2 \pmod 5$

$$\therefore \quad \begin{aligned} v_1 &= v_6 = v_{11} = \ldots = 1 \\ v_2 &= v_7 = v_{12} = \ldots = 0 \\ v_3 &= v_3 = v_{13} = \ldots = 1 \\ v_4 &= v_9 = v_{14} = \ldots = 1 \\ v_5 &= v_{10} = v_{15} = \ldots = 0 \end{aligned}$$

Thus, the required period may be written

$$\overset{\times}{1}011\overset{\times}{0}$$

3. It is important to note that the period associated with $\left[\frac{an+b}{p}\right]$ where a and b are less than p, depends only upon $\frac{a}{p}$ and

$\left[\frac{b}{h}\right]$ where h is the highest common factor of a and p. Therefore, we have the following theorem :

Theorem I. *The necessary and sufficient conditions for the equality*

$$\left[\frac{an+b}{p}\right]=\left[\frac{cn+d}{q}\right]$$

are

$$\text{(i)}\quad \frac{a}{p}=\frac{c}{q}, \quad \textit{and} \quad \text{(ii)}\quad \left[\frac{b}{h}\right]=\left[\frac{d}{j}\right]^{*}$$

where a b, p, c, d, q *are all positive integers and* h, j *are the highest common factors of* (a, p), (c, q) *respectively.*

The above theorem is true even when a and b are not less than p and c and d are not less than q. For, suppose $a > p$ and $b > v$; then

$$u_n=\left[\frac{an+b}{p}\right]=n\left[\frac{a}{p}\right]+\left[\frac{b}{p}\right]+\left[\frac{a'n+b'}{p}\right]$$

where a', b' are respectively the remainders obtained by dividing a and b by p, and $v_{n+1}=\left[\frac{a}{p}\right]$ + 0 or 1 according as

$$\left.\begin{array}{c}\frac{a'}{h}\cdot n \equiv -\left[\frac{b'}{h}\right], -\left[\frac{b'}{h}\right]+1, \ldots \\ \text{or} \\ -\left[\frac{b'}{h}\right]-\frac{a'}{h}, \ldots\end{array}\right\}\left(\bmod \frac{p}{h}\right)$$

where h is the greatest common divisor of a' and p and therefore also of a and p.

Hence the period of $\left[\frac{an+b}{p}\right]$ is obtained from that of $\left[\frac{a'n+b'}{p}\right]$ by increasing each element of the latter period by $\left[\frac{a}{p}\right]$. For example, the elements of the period of $\left[\frac{52n+73}{20}\right]$ diminished by 2 are the

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* $\left[\frac{b}{h}\right]=\left[\frac{d}{j}\right]$ implies $\left[\frac{b}{p}\right]=\left[\frac{d}{q}\right]$ since $\frac{p}{h}=\frac{q}{j}$ = an integer ; hence the identity is true when $n = 0$ and therefore also when n = 1, 2, 3, 4, etc. from the identity of the periods.

same as that of $\left[\frac{9n + 11}{15}\right]$ when each element of the latter is increased by $\left[\frac{52}{20}\right]$ or 2, since

$$\left[\frac{52n + 73}{20}\right] = 2n + 3 + \left[\frac{12n + 13}{20}\right]$$

$$= 2n + 3 + \left[\frac{9n + 11}{15}\right]$$

$$\because \qquad \frac{12}{20} = \frac{9}{15} \text{ and } \left[\frac{13}{4}\right] = \left[\frac{11}{3}\right]$$

Thus, the period of $\left[\frac{52n + 73}{20}\right]$ may be written

$$\overset{\times}{3}23\overset{\times}{3}2$$

4. The particular case, where $a = 1$ and $b < p$, deserves mention.

The period of $\left[\frac{n + b}{p}\right]$ contains p elements, of which the element in the $(p - b)$th place is 1 and the other elements are zero; *i.e.* the *unit* elements occur in the places

$$(p - b)\text{th}, (2p - b)\text{th}, \ldots$$

Since these numbers are the indices of the powers of x in $x^{p-b}/(1 - x^p)$, this expression may formally represent not only the period of $\left[\frac{n + b}{p}\right]$ but also $\left[\frac{n + b}{p}\right]$ itself.

The advantage of this representation is that any set of identities involving addition and subtraction of expressions of the type $\left[\frac{n + b}{p}\right]$ exactly corresponds to algebraic identities involving similarly expressions of the type $x^{p-b}/(1 - x^p)$, and there is one-to-one correspondence between the two sets of identities, one arithmetical and the other algebraical. Thus we may paraphrase a set of arithmetical identities into a set of algebraic ones and *vice-versâ* and this method may be called *the method of paraphrase*. The scope of this method is illustrated in the following theorem:

5. *Theorem* II. For all positive integral values of n,

$$\left[\frac{n}{p}\right] = \left[\frac{n}{pq}\right] + \left[\frac{n + p}{pq}\right] + \ldots \ldots + \left[\frac{n + p(q - 1)}{pq}\right]$$

p, q being any two positive integers.

Now, $\left[\frac{n + rp}{pq}\right]$, where $r < q$ may be represented by the algebraic expression $x^{p(q-r)}/(1 - x^{pq})$.

Therefore, the expression on the right-hand side of the theorem can be represented by

$$\frac{x^{pq} + x^{p(q-1)} + x^{p(q-2)} + \ldots\ldots + x^{p(q-r)} + \ldots\ldots + x^{p}}{1 - x^{pq}}$$

which is equal to

$$\frac{x^{p}}{1 - x^{p}}$$

This last expression being the representation of $\left[\frac{n}{p}\right]$, the truth of the theorem becomes apparent.

Cor. (1) $$n = \left[\frac{n}{q}\right] + \left[\frac{n+1}{q}\right] + \left[\frac{n+2}{q}\right] + \ldots\ldots + \left[\frac{n+q-1}{q}\right]$$

where q is any positive integer (not zero).

Cor. (2) Changing in (1) n to $(an + b)$ where a, b are any two integers, we get another identity

$$an + b = \left[\frac{an+b}{q}\right] + \left[\frac{an+b+1}{q}\right] + \ldots + \left[\frac{an+b+q-1}{q}\right]$$

Cor. (3) Since

$$\left[\frac{n}{q}\right] = \left[\frac{n}{pq}\right] + \left[\frac{n+q}{pq}\right] + \ldots + \left[\frac{n+q(p-1)}{pq}\right],$$

we get $$\left[\frac{n}{p}\right] + \Sigma\left[\frac{n+rq}{pq}\right] = \left[\frac{n}{q}\right] + \Sigma\left[\frac{n+sp}{pq}\right]$$

where r takes all non-zero positive integral values less than p except those which are multiples of p/h, and s take similarly all non-zero positive integral values less than q except those which are multiples of q/h, (h being the greatest common divisor of p and q).

For example when $p = 12$, $q = 8$, we have the identities

$$\left[\frac{n}{12}\right] = \left[\frac{n}{96}\right] + \left[\frac{n+12}{96}\right] + \left[\frac{n+24}{96}\right] + \left[\frac{n+36}{96}\right] + \left[\frac{n+48}{96}\right] + \left[\frac{n+60}{96}\right] + \left[\frac{n+72}{96}\right] + \left[\frac{n+84}{96}\right]$$

$$\left[\frac{n}{8}\right] = \left[\frac{n}{96}\right] + \left[\frac{n+8}{96}\right] + \left[\frac{n+16}{96}\right] + \left[\frac{n+24}{96}\right]$$

$$+ \left[\frac{n+32}{96}\right] + \left[\frac{n+40}{96}\right] + \left[\frac{n+48}{96}\right] + \left[\frac{n+56}{96}\right]$$

$$+ \left[\frac{n+64}{96}\right] + \left[\frac{n+72}{96}\right] + \left[\frac{n+80}{96}\right] + \left[\frac{n+88}{96}\right]$$

$$\therefore \left[\frac{n}{12}\right] + \left[\frac{n+8}{96}\right] + \left[\frac{n+16}{96}\right] + \left[\frac{n+32}{96}\right] + \left[\frac{n+40}{96}\right]$$

$$+ \left[\frac{n+56}{96}\right] + \left[\frac{n+64}{96}\right] + \left[\frac{n+80}{96}\right] + \left[\frac{n+88}{96}\right]$$

$$= \left[\frac{n}{8}\right] + \left[\frac{n+12}{96}\right] + \left[\frac{n+36}{96}\right] + \left[\frac{n+60}{96}\right] + \left[\frac{n+84}{96}\right]$$

Changing n to $4n$, we derive the simpler identity

$$\left[\frac{n}{3}\right] + \left[\frac{n+2}{24}\right] + \left[\frac{n+4}{24}\right] + \left[\frac{n+8}{24}\right] + \left[\frac{n+10}{24}\right]$$

$$+ \left[\frac{n+14}{24}\right] + \left[\frac{n+16}{24}\right] + \left[\frac{n+20}{24}\right] + \left[\frac{n+22}{24}\right]$$

$$= \left[\frac{n}{2}\right] + \left[\frac{n+3}{24}\right] + \left[\frac{n+9}{24}\right] + \left[\frac{n+15}{24}\right]$$

$$+ \left[\frac{n+21}{24}\right]$$

which may also be obtained more directly by using

$$\left[\frac{n}{3}\right] = \left[\frac{n}{24}\right] + \left[\frac{n+3}{24}\right] + \dots$$

and

$$\left[\frac{n}{2}\right] = \left[\frac{n}{24}\right] + \left[\frac{n+2}{24}\right] + \dots$$

Putting $p = 2$, $q = 3$ in Cor. (3) above, we get

$$\left[\frac{n}{3}\right] + \left[\frac{n+2}{6}\right] + \left[\frac{n+4}{6}\right] = \left[\frac{n}{2}\right] + \left[\frac{n+3}{6}\right]$$

which is Ramanujan's identity.

6. *Theorem* III. *The necessary and sufficient condition for the equality* $[(an+b)^{1/p}] = [(cn+d)^{1/p}]$, *where* a, b, c, d, p *are all positive integers and* $b > d$ *to hold for all positive integral values of* n, *including zero, are*

(i) $$a = c$$

and (ii) *the integers* $d+1, d+2, \ldots b$ *should be* p-*ic non=residence of* a.

If $a \neq c$, then it is easily seen that

$$(an+b)^{1/p} \gtrless (cn+d)^{1/p}$$

according as $a \gtrless c$, for all values of $n > \dfrac{b-d}{c-a}$.

Since $$[(an+b)^{1/p}] = [(cn+d)^{1/p}],$$

we have

$$(an+b)^{1/p} \backsim (cn+d)^{1/p}.$$

But $$\underset{n\to\infty}{\text{Lt.}} \left(\frac{an+b}{cn+d}\right)^{1/p} = \left(\frac{a}{c}\right)^{1/p}$$

Hence, $$\left(\frac{a}{c}\right)^{1/p} = 1, \text{ i.e. } a = c.$$

This is our first condition.

Now, if $(an+b)$ is a perfect pth power, say k^p: where k is an integer, then

$$(an+b)^{1/p} = k > (an+d)^{1/p}, \text{ so that}$$

$$[(an+b)^{1/p}] > [(an+d)^{1/p}].$$

Similarly, if $an+e$ is a perfect pth power, where $b > e > d$, we have

$$[(an+b)^{1/p}] > [(an+d)^{1/p}].$$

Therefore, the necessary condition for

$$[(an+b)^{1/p}] = [(an+d)^{1/p}] \quad (b > d)$$

is that, for no value of n, can $an+e$, where e is an integer such that $b \geqslant e > d$, be a perfect pth power.

In other words, $d+1, d+2, \ldots (b-1), b$ should all be p-ic non-residues of a.

Conversely, when $an + e$ is not a perfect p-th power e being any integer such that $b \geqslant e > d$, it is evident that it should lie between the p-th powers of two consecutive integers k, $k + 1$ (say)

i.e. $$(k + 1)^p > (an + e) > k^p.$$

Thus, $k + 1 > (an + b)^{1/p} > (an + b - 1)^{1/p} > \dots >$

$$(an + d + 1)^{1/p} > (an + d)^{1/p} \geqslant k$$

Hence $$[(an + b)^{1/p}] = [(an + d)^{1/p}]$$

This proves that the given conditions are also sufficient.

***Cor*:** If r, s be any two positive integers, then

$$\left[\frac{r + (an + b)^{1/p}}{s}\right] = \left[\frac{r + (an + d)^{1/p}}{s}\right],$$

provided a, b, d are all positive integers and the integers (if any) between b and d as well as the greater number of the pair (b, d) are p-ic non-residues of a.

Setting $r = d = 1$, $s = b = p = 2$, and $a = 4$, we get Ramanujan's result,

$$[\tfrac{1}{2} + \sqrt{(n + \tfrac{1}{2})}] = [\tfrac{1}{2} + \sqrt{(n + \tfrac{1}{4})}]$$

As another example, we may mention

$$\left[\frac{1}{2} + \sqrt[5]{n + \frac{15}{32}}\right] = \left[\frac{1}{2} + \sqrt[5]{n + \frac{1}{2}}\right];$$

since all the even numbers (< 32) are the 5-ic non-residues of 32 while all the odd numbers (< 32) are the 5-ic residues.

More generally, when p is any integer,

$$\left[\frac{1}{2} + \left(n + \frac{q}{2^p}\right)^{1/p}\right] = \left[\frac{1}{2} + \left(n + \frac{q + 1}{2^p}\right)^{1/p}\right]$$

where q is an odd integer less than 2^p.

ON DR. MARR'S DESARGUES CONFIGURATION.

By B. Ramamurti,
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1. In a paper published in the *Journal of the London Math. Soc.* **7** (1926), pp. 86—93, Dr. Marr has shown that given six points on a conic there are ten points P such that the lines joining P to the six points touch a cubic curve at these six points. Recently in an additional note* he has shown that these points form a Desargues configuration. He has not stated that this configuration is self-polar with respect to the conic, though this follows from his proof.

2. But six points on a conic determine uniquely a self-polar Desargues Configuration in the following manner. It is well-known that given six points on a conic, there are five pencils of quartics having these as Jacobian points. Taking any one of them, there are three harmonic quartics (*i e* quartics breaking up into two apolar quadratics) in it. The joins of the pairs of points correspondiag to the three pairs of apolar quadratics form the sides of a self-polar quadrangle (*i e.* each vertex is the centre of perspective of the triangle formed by the other three and its polar triangle). The pencil of conics circumscribing the quadrangle cut out on the base conic the pencil of quartics we have taken. This quadrangle together with its polar quadrilateral constitutes a self-polar Desargues Configuration of ten points and ten lines, out of which there can be formed five self-polar quadrangles. These five quadrangles determine the five co-Jacobian pencils of quartics on the base conic.† Every one of these pencis leads to the same configuration, which may hence be called the Jacobian Desargues Configuration of the six points The object of this note is to show that Dr. Marr's Configuration is none other than the Jacobian Desargues Configuration of the six points.

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* *Journal, London Math Soc.*, 5 (1930), pp. 193—95.

† Stephanos : *Mem de l' Institute (1883; 2.*

3. I make use of the following result of Dr. Marr.

If the polar of P (one of the points under investigation) with respect to the conic cuts it at Q and R, Q and R are polar points of each other with respect to the binary sextic constituded by the given six points. By retracing the steps of his proof, the converse also can be shown to be true. ... (1)

I make use also of the following second result of Dr, Marr, for which I give a different proof.

Given six points on a conic there are ten different positions of P. ... (2)

A point and its polar points with respect to a sextic determine an algebraic (1, 5) correspondence, whose fixed points are the points of the sextic. Hence a point and the polar points of its polar points determine a (1,25) correspondence whose fixed points are 26 of which six are given by the sextic. It can be easily shown that the remaining 20 occur in ten pairs of mutually polar points. For if Q be a fixed point, it means that of its five polar points there is one R for which Q is a polar point. Hence R also is a fixed point and Q and R are polar points of each other. Hence taking the pole P of QR, there are ten different positions of P, from (1).

4. I shall now prove symbolically* that every line of the Jacobian Desargues configuration cuts the conic in a pair of points mutually polar with respect to the Jacobian sextic. We know that every line of the configuration cuts the conic in a pair of points corresponding to one of the mutually apolar quadratic factors of a harmonic quartic of one of the five co-jacobian pencils.

If the two points be given by

$$(\alpha x) = (\alpha_1 x_2 - \alpha_2 x_1) = 0 \text{ and } (\beta x) = (\beta_1 x_2 - \beta_2 x_1) = 0$$

the other pair can be given by

$$(\alpha x)^2 + (\beta x)^2 = 0$$

whence the pencil can be written as

$$(\alpha x)(\beta x)\{(\alpha x)^2 + (\beta x)^2\} + \lambda (\gamma x)^4 = 0$$

where $(\gamma x)^4 = 0$ represents symbolically a general quartic of the pencil.

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* I am indebted to Dr. R. Vaidyanathaswamy for suggesting the method for the symbolic proof.

The Jacobian is $J = A + B$

where $$A = \{ (\alpha x)^3 (\beta x), (\gamma x)^4 \}^1$$

and $$B = \{ (\beta x)^3 (\alpha x), (\gamma x)^4 \}^1$$

Lemma. α and β are polar points of each other with respect to $A = 0$.

For $$A = (\gamma x)^3 \{ 3 (\alpha x)^2 (\beta x) (\alpha \gamma) + (\alpha x)^3 (\beta \gamma) \}$$

Since this involves the factor $(\alpha x)^2$, α is a polar point of every other point and hence in particular of β.

Polarising A with respect to α, we have

$$3 (\gamma x)^2 (\gamma \alpha) \{ 3 (\alpha x)^2 (\beta x) (\alpha \gamma) + (\alpha x)^3 (\beta \gamma) \}$$
$$+ (\gamma x)^3 \{ 3 (\alpha x)^2 (\beta \alpha) (\alpha \gamma) \}$$

Substituting β for x, this vanishes.

Hence β is a polar point of α.

Therefore α and β are polar points of each other with respect to $A = 0$.

From symmetry it follows that α and β are polar points of each other with respect to $B = 0$ and hence with respect to the six points given by $J = A + B = 0$.

5. Thus the ten lines of the Jacobian configuration cut the conic in pairs of mutually polar points and from (2) there can be only ten such pairs. Hence the ten poles of these lines, which are but the vertices of the same Jacobian Desargues configuration are from (1), the ten positions of the point P of Dr. Marr.

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NOTES AND QUESTIONS

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Notes and Questions.

A construction for the double points of an involution in the Gauss plane and its applications.

In § 1 of this note, the construction is given. In § 2 this is generalised so as to give an inversionally invariant construction for the mate of any point, when two pairs of an involution are given. Also from this is deduced a necessary and sufficient condition for three pairs of points to belong to an involution. Incidentally a fourth pair belonging to the above involution is obtained, and the relation between the four pairs is studied in § 3.

1. The construction required is equivalent to Prob. 3, Chap. VIII, Coolidge: *Treatise on the Circle and the Sphere* :—"Given two pairs of points of a Möbius involution to find the double points." Since the author aims at the invariance of each step for inversion the construction is long, and in the end the double points are not definitely obtained. They are either the points of intersection of a co-axal system of circles or the limiting points as the case may be. Here a simple construction is given, though the steps are not invariant for inversion.

Given the pairs A_1A_2, B_1B_2, let A_1B_1, A_2B_2, intersect at L_1 and circles $L_1A_1B_2$, $L_1A_2B_1$ at O_1. On either side of the bisector of angle $A_1O_1A_2$ take P and Q such that $O_1P^2 = O_1Q^2 = O_1A_1 . O_1A_2$.

P and Q are the required points.

For, it can be easily shown that triangles $O_1A_1B_1$ and $O_1A_2B_2$ are similar whence $O_1A_1 . O_1A_2 = O_1B_1 . O_1B_2 = O_1P^2 = O_1Q^2$ and PQ bisects also $\angle B_1O_1B_2$.

Hence PQ separate harmonically both A_1A_2 and B_1B_2. So they are the double points.*

Similarly let A_1B_2, A_2B_1 intersect at L_2. Then circles $L_2A_1B_1$ and $L_2A_2B_2$ will intersect again at O_1.

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* Grace and Young : *Algebra of Invariants*, page 209.

2. Since O_1 is the centre of involution it is the mate of O_2 the single point at infinity in the Gauss plane. The construction is generalised by taking any point C_1 instead of the point at infinity, so as to give the mate of C_1.

Construction. Let circles $C_1A_1B_1$, $C_1A_2B_2$, intersect at D_1. Then circles $D_1A_1B_2$, $D_1A_2B_1$ will intersect at C_2 the mate of C_1.

Proof. Invert with respect to C_1. C_1 inverts into the point at infinity O_2, D_1 into L_1 and C_2 into O_1.

But O_1 is the mate of O_2. Since involution property is invariant for inversion C_2 is the mate of C_1.

We may take circles $C_1\ A_1\ B_2$ and $C_1\ A_2\ B_1$ intersecting at D_2. Then it can be shown similarly that circles $D_2\ A_1\ B_1$ and $D_2\ A_2\ B_2$ intersect again at C_2. From the above may be deduced the following theorem :—

A necessary and sufficient condition for three pairs $A_1\ A_2$, $B_1\ B_2$, $C_1\ C_2$ to belong to an involution is that the circles $C_1\ A_1\ B_1$, $C_1\ A_2\ B_2$, $C_2\ A_1\ B_2$, $C_2\ A_2\ B_1$ should intersect at a common point D_1... (1)

From the argument stated above it follows also that the circles,

$C_1\ A_1\ B_2$, $C_1\ A_2\ B_1$, $C_2\ A_1\ B_1$, $C_2\ A_2\ B_2$ will intersect at a common point D_2. ... (2)

The circles of the system (1) are identical if $A_1\ A_2$, $B_1\ B_2$, $C_1\ C_2$ lie on a circle. In this case it is known that the necessary and sufficient condition is that $A_1\ A_2$, $B_1\ B_2$, and $C_1\ C_2$ should be concurrent.

3. The pair of points $D_1\ D_2$ also belong to the above involution. For from (1) and (2) it follows that circles $D_1\ A_1\ B_1$, $D_1\ A_2\ B_2$, $D_2\ A_1\ B_2$, $D_2\ A_2\ B_1$ intersect at C_1 and circles $D_1\ A_1\ B_2$, $D_1\ A_2\ B_1$ $D_2\ A_1\ B_1$, $D_2\ A_2\ B_2$ intersect at C_2. Hence $D_1\ D_2$ belong to the above involution.

As a particular case, if $A_1\ B_1$, $A_2\ B_2$ intersect at L_1 and $A_1\ B_2$, $A_2\ B_1$ at L_2, $L_1\ L_2$ belong to the involution.

For the pair $C_1\ C_2$ let us take a double point P reckoned twice. Since P is its own mate, it follows from (1) and (2) that D_1 and D_2 should coincide with P. Hence we have this invariant property :—If P

is a double point of the involution determined by A_1 A_2 and B_1 B_2, the circles PA_1 B_1 and PA_2 B_2 touch at P, as also the circles PA_1 B_2 and PA_2 B_1.

The mutual relation of the four point-pairs is symmetrical. Given any three non-cyclic pairs, the fourth one is uniquely determined. Now the point pairs of an involution are given by a pencil of quadratics in the complex variable z. Here I show that the cross-ratio of the parameters of the four point-pairs is real, or in other words, the points in the Gauss plane corresponding to these lie on a circle. In this case it will follow that the mid-points of the four point-pairs lie on a circle and that D_1 D_2 lie on the unique sub-rational bicircular quartic defined by A_1 A_2, B_1 B_2 and C_1 C_2.

Proof. Consider a linear transformation in the complex variable which carries the double-points of the above involution into the origin and the single point at infinity. Then the four pairs will be given by

$$z^2 - \mu = 0 \qquad \mu = \mu_1,\ \mu_2,\ \mu_3 \text{ or } \mu_4.$$

$$A_1 = \sqrt{\mu_1},\ A_2 = -\sqrt{\mu_1} \text{ and so on.}$$

From (1) since C_1 A_1 D_1 B_1 are cyclic the cross ratio

$$(\sqrt{\mu_3}\ \sqrt{\mu_1}\ \sqrt{\mu_4}\ \sqrt{\mu_2})$$

is real or

$$\frac{(\sqrt{\mu_3}-\sqrt{\mu_1})(\sqrt{\mu_4}-\sqrt{\mu_2})}{(\sqrt{\mu_3}-\sqrt{\mu_2})(\sqrt{\mu_4}-\sqrt{\mu_1})}$$

is real. Also since C_1 A_2 D_1 B_2 are cyclic

$$\frac{(\sqrt{\mu_3}+\sqrt{\mu_1})(\sqrt{\mu_4}+\sqrt{\mu_2})}{(\sqrt{\mu_3}+\sqrt{\mu_2})(\sqrt{\mu_4}+\sqrt{\mu_1})}$$

is real. Multiplying these two

$$\frac{(\mu_3-\mu_1)(\mu_4-\mu_2)}{(\mu_3-\mu_2)(\mu_4-\mu_1)}$$

is real, whence the points represented by μ_1, μ_2, μ_3, μ_4 are concyclic.

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On the intersections of Three Planes.

The discussion of this topic in article 45 of Bell's *Solid Geometry* is inaccurate and incomplete; the following note is prepared with a view to correct it and supplement it where necessary.

The notation in Bell's book is adhered to in what follows. Besides, however, the symbols and expressions already used there, the following additional expressions are used here. The nine minors of the determinant $| a_1 b_2 c_3 | (\equiv \Delta)$ are classified in two ways—according to the sameness of (i) the letter and (ii) the suffix—into three sets, each set containing three minors. Thus A_1, A_2, A_3: B_1, B_2, B_3; and C_1, C_2, C_3, are called respectively the A-series, the B-series, and the C-series: and A_1, B_1, C_1; A_2, B_2, C_2; and A_3, B_3, C_3 are called respectively the 1st series, the 2nd series, and the 3rd series.

The following propositions will be found useful in what follows:—

Proposition I. If $\Delta = 0$, and any one of the nine minors, $A_r\ B_r\ C_r$ $(r = 1, 2, 3)$, say A_1, is zero; then each member of *either* of the two series corresponding to it vanishes, *i e*, in the particular instance chosen, *either* $A_1 = A_2 = A_3 = 0$ *or* $A_1 = B_1 = C_1 = 0$.

Proof. We have got the following identities:—

$$C_1A_2 - C_2A_1 = b_3 \Delta$$
$$C_3A_1 - C_1A_3 = b_2 \Delta$$
$$A_1B_2 - A_2B_1 = c_3 \Delta$$
$$A_3B_1 - A_1B_3 = c_2 \Delta.$$

As $\Delta = A_1 = 0$, these identities reduce to

$$C_1A_2 = 0$$
$$C_1A_3 = 0$$
$$A_2B_1 = 0$$
$$A_3B_1 = 0.$$

These latter four equations are equivalent to *either* $A_2 = A_3 = 0$, *or* $B_1 = C_1 = 0$. Hence we get either $A_1 = A_2 = A_3 = 0$, or

$A_1 = B_1 = C_1 = 0$ as a necessary consequence of the simultaneous vanishing of Δ and A_1.

Proposition II. If $\Delta = 0$, and $A_1 = A_2 = 0$, then *either* $A_3 = 0$, *or* $A_1 = A_2 = B_1 = B_2 = C_1 = C_2 = a_3 = b_3 = c_3 = 0$.

Proof. By the application of the preceding proposition, because $\Delta = A_1 = 0$, *either* $A_1 = A_2 = A_3 = 0$ *or* $A_1 = B_1 = C_1 = 0$. If we take the first alternative, then the additional fact known to us, namely, $A_2 = 0$, does not *of necessity* lead us to any other conclusion besides $A_1 = A_2 = A_3 = 0$ If, however, we take the second alternative, then the additional fact of A_2 being equal to zero necesarily leads us to $A_2 = B_2 = C_2 = 0$. It is assumed that when the second alternative is taken, the first does not hold good. We, therefore, come in the event of the second alternative to the result $A_1 = B_1 = C_1 = A_2 = B_2 = C_2 = 0$, and we know $A_3 \neq 0$, the first alternative not holding good. Also we have the following three relations :—

$$a_1A_1 + a_2A_2 + a_3A_3 = \Delta = 0 = a_3A_3 = a_3$$

$$b_1A_1 + b_2A_2 + b_3A_3 \qquad = 0 = b_3A_3 = b_3$$

$$c_1A_1 + c_2A_2 + c_3A_3 \qquad = 0 = c_3A_3 = c_3 \, (A_1 = A_2 = 0 \, ; \, A_3 \neq 0).$$

Thus finally in the case of the second alternative, we come to the result $A_1 = A_2 = B_1 = B_2 = C_1 = C_2 = a_3 = b_3 = c_3 = 0$.

We may generalise this proposition as follows :—"If any two members of the same letter (or number) series vanish in addition to Δ being equal to zero, then *either* the remaining member of that letter (or number) series is equal to zero, or the four remaining members of the two number (or letter) series corresponding to the two assumed vanishing members together with the three small letters corresponding to the third number (or letter) series vanish each separately."

Bell has considered in his treatment three cases :—(I) $\Delta \neq 0$, (II) $\Delta = 0$ and $| b_1c_2d_3 | \neq 0$, and (III) $\Delta = 0$, $| b_1c_2d_3 | = 0$ and $A_1 \neq 0$, (A_1 is one of the common minors of Δ and $| b_1c_1d_3 |$). The

treatment does not clearly bring out the fact that the three cases, rightly interpreted, exhaust all possibilities except one. The hypothesis in the second case, *viz.*, $\Delta = 0$ and $| b_1\ c_2\ d_3 | \neq 0$ is to be interpreted to mean that " $\Delta = 0$ and at least one of $| b_1\ c_2\ d_3 |$, $| a_1\ c_2\ d_3 |$ and $| a_1\ b_2\ d_3 |$ is not equal to zero." If this interpretation is put upon the hypothesis in case II, the only possibility left after cases I and II have been considered is that $\Delta = 0$ and each one of $| b_1\ c_2\ d_3 |$, $| a_1\ c_2\ d_3 |$ and $| a_1\ b_2\ d_3 |$ is zero. This possibility is furthur divided into two cases—one in which at least one of the common minors of any one of the three latter determinants and Δ is not zero (case III in Bell's treatment) and the other in which all the common minors of Δ and the three latter determinants are each equal to zero. This last case, which we may call case IV, has been omitted by Bell, probably because it leads to the obvious geometrical conclusion that the three planes are parallel. That the description of case III given by Bell covers the whole field required to be covered by it in the above detailed classification is evident, for the four determinants Δ, $| b_1\ c_2\ d_3 |$, $| a_1\ c_2\ d_3 |$ and $| a_1\ b_2\ d_3 |$ are such that if any two of them vanish, and one of their common minors does not vanish, then the remaining two determinants also vanish. (*Vide* the footnote on p. 51 in Bell's book). It is, therefore, necessary to interpret the hypothesis in case II as " $\Delta = 0$ and at least one of $| b_1\ c_2\ d_3 |$, $| a_1\ c_2\ d_3 |$, $| a_1\ b_2\ d_3 |$ is not zero " in order to bring out clearly the fact that Bell's classification can be made exhaustive with the addition of case IV.

Case II, therefore, in Bell's treatment as interpreted above can occur in any one of the following three ways:—

1st type of Case II.

(i) $\Delta = 0$, and each of $| b_1\ c_2\ d_3 |$, $| a_1\ c_2\ d_3 |$, and $| a_1\ b_2\ d_3 |$ is different from zero. It should be noted that $| b_1\ c_2\ d_3 |$, $| a_1\ c_2\ d_3 |$ and $| a_1\ b_2\ d_3 |$ are respectively the same as $d_1A_1 + d_2A_2 + d_3A_3$. $-(d_1B_1 + d_2B_2 + d_3B_3)$, $d_1C_1 + d_2C_2 + d_3C_3$; and each of these is different from zero. Now if any one of the nine minors of Δ, say A_1, is zero, then by proposition I either $A_1 = A_2 = A_3 = 0$ or $A_1 = B_1 = C_1 = 0$, because simultaneously with A_1, $\Delta = 0$. The first alterna-

tive, however, cannot hold good since in that case $d_1A_1 + d_2A_2 + d_3A_3$ shall have to become equal to zero. It therefore follows that $A_1 = B_1 = C_1 = 0$ is the only possible alternative. If in addition to this (*i.e.*, $A_1 = 0$) any member of a different numerical series, say B_2, is zero, that will similarly lead to $A_2 = B_2 = C_2 = 0$. But then, as $A_1 = B_1 = C_1 = A_2 = B_2 = C_2 = 0$, but none of A_3, B_3, C_3 is zero, it follows by proposition II that $a_3 = b_3 = c_3 = 0$ additionally. Thus this sub-case (i) under case II can occur in the following three ways:—

(*a*) $\Delta = 0$, none of $| b_1 \, c_2 \, d_3 |$, $| a_1 \, c_2 \, d_3 |$ and $| a_1 \, b_2 \, d_3 |$ is zero, and none of A_r, B_r, C_r $(r = 1, 2, 3)$ is zero. The three planes in this case form a triangular prism, no two of them being parallel,

(*b*) $\Delta = 0$, none of $| b_1 \, c_2 \, d_3 |$, $| a_1 \, c_2 \, d_3 |$, and $| a_1 \, b_2 \, d_3 |$ is zero, and minors of only one numerical series, say A_1, B_1, C_1, out of the nine minors of Δ are each equal to zero. It is clear that the planes $u_2 = 0$ and $u_3 = 0$ are parallel because $A_1 = B_1 = C_1 = 0$. The three planes, therefore, form a triangular prism, one of whose edges is at infinity, *i.e.*, a set of two parallel planes intersected by the third.

[NOTE.—The statement in lines 1 to 4 on page 50 in Bell's book, *viz.*, "If two or more ... ... a triangular prism," is partially incorrect: since it is quite possible for each of A_2, A_3 not to become equal to zero, and for each of A_1, B_1, C_1 to be equal to zero thus making the planes $u_2 = 0$ and $u_3 = 0$ parallel. This statement can obviously be corrected by substituting the word "all" for the words "two or more" The next statement in lines 4 to 7, *viz.* "If one only ... in parallel lines," is also incorrect; for if $A_1 \neq 0$; and $A_2 = A_3 = 0$, then as $\Delta = 0$ it follows by proposition II that $A_2 = A_3 = B_2 = B_3$ $C_2 = C_3 = a_1 = b_1 = c_1 = 0$; and the plane $u_1 = 0$, therefore, coincides with the plane at infinity. This statement can be corrected by putting it as follows:—"If only, A_3, say, of the three quantities is zero, the planes $u_1 = 0$ and $u_2 = 0$ are parallel, and $u_3 = 0$ meets them in parallel lines."]

(*c*) $\Delta = 0$, none of $| b_1 c_2 d_3 |$, $| a_1 c_2 d_3 |$, and $| a_1 b_2 d_3 |$ is zero, and minors of only two numerical series, say A_1, B_1, C_1, A_2, B_2, C_2, out

of the nine minors of Δ are each equal to zero. Under these circumstances $a_3 = b_3 = c_3 = 0$ as has already been shown. Here the plane $u_3 = 0$ coincides with the plane at infinity, and thus the three planes form a triangular prism two of whose edges are at infinity.

[NOTE —The line through the origin, which is parallel to the finitely placed edges (or 'edge' as in (*c*)} of the prism formed by the three planes, lies in none of the co-ordinate planes. This is the distinguishing characteristic of the first type of Case II.]

2nd type of Case II.

(ii) $\Delta = 0$ and any two only of the remaining three determinants are different from zero, say, $| b_1c_2d_3 | \neq 0$, and $| a_1c_2d_3 | \neq 0$. The third determinant is $| a_1b_2d_3 |$, *i.e.*, $d_1C_1 + d_2C_2 + d_3C_3$, and this alone is equal to zero together with Δ. It follows, therefore, as a consequence of the proposition proved in the foot-note on page 51, that each of C_1, C_2, C_3 is zero. Except for this additional property—which leads to the result that the line through the origin parallel to the finitely-placed edges (or edge) of the prism formed by the three planes lies in the co-ordinate plane $z = 0$—the discussion for this sub-case (ii) is quite the same as that for sub-case (i).

Third type of Case II.

(iii) $\Delta = 0$ and any one only of the remaining three determinants is different from zero, say $| b_1c_2d_3 | \neq 0$. As $| a_1c_2d_3 |$, $| a_1b_2d_3 |$, and Δ only vanish, it follows as in the preceding sub-case and by proposition II that $B_1 = B_2 = B_3 = C_1 = C_2 = C_3 = a_1 = a_2 = a_3 = 0$, and thus each of the three planes becomes parallel to the x-axis, the line through the origin parallel to the finitely-placed edges (or edge) of the prism formed by the three planes lying in both the co-ordinate planes $y = 0$ $z = 0$, *i.e.*, coinciding with the x-axis. Except for this property, the discussion for this sub-case is just the same as that for the preceding two.

G. S. DIWAN.

Marginal notes on Volume II of Smith's History of Mathematics.

The sub-title of this volume is "Special Topics of Elementary Mathematics," and the sub-title of Volume I is "General Survey of the History of Elementary Mathematics." Hence it would have been consistent to include the word 'elementary' in the title of the entire work, and this title would then have been more appropriate. In view of the fact that this work will probably be consulted for many years, it seems likely that some owners thereof may find it desirable to add in their own copies references to places where corrections or other useful modifications may be found. The following notes aim to be especially useful to such readers, but they are also intended to assist those who desire to form an independent judgment as regards the trustworthiness of this unique and interesting work relating to the history of elementary mathematics.

On page 13 there appears the following sentence: "Since unity was not considered a number until modern times, it was sometimes definitely omitted, leaving eight digits." The equivalent of the first part of this quotation appears also on page 26 where it is expressed as follows: "Not until modern times was unity considered a number." These quotations are very striking and naturally give rise to various queries as regards the intellectual development of the human race before the period which is commonly denoted by the term "modern times." It is well-known that people counted in very early times and that in answering the question "how many?" the number one, or unity, would naturally be regarded as a number just as two, three, etc., were regarded as numbers. In the numerical notations of various ancient people unity was represented by a single stroke, while two, three, etc., were often represented by a number of strokes corresponding to the number of units up to the number ten.

Hence it is clearly somewhat difficult to harmonize the given quotations and some well-established facts. This difficulty is likely to be increased if other authorities are consulted. For instance, if one consults J. Tropfke's noted *Geschichte der Elementar-Mathematik*, one finds on page 54, Volume 2 (1921), that while unity was not at first regarded as a number by the ancient Greeks, it gradually secured for itself among the Greek mathematicians its proper position in their number system. In particular, Theon of Smyrna (about 130) stated at one place that unity is not a number, only the source of numbers, but he later included unity

among the natural numbers. In Book VII of his *Elements*, Euclid defined a number as a multiple composed of units and hence it is easy to see why so many later writers adopted the view that unity should not be regarded as a number.

The unique properties of unity, such as the fact that every natural number after the first can be obtained by adding unity to the preceding natural number and that a number is not changed when it is multiplied by unity, probably did not impress the people forcibly until they had reached a somewhat advanced stage of intellectual development. This hypothesis is in accord with the fact that the ancient Greeks and the ancient Hindus seem to have been the first to question the desirability of including unity among the natural numbers. It should be emphasized that it cannot be proved that unity is a number, since this depends entirely upon the definition of the term number. It is, however, not true that the term number was always used up to modern times with a definition, either explicit or implicit, which excluded unity, and hence the quotations noted above are apt to convey an incorrect impression.

On page 21 it is stated that "Fibonacci (1202) gave $\frac{1}{2} \cdot 2^2 (2^2 - 1) = 6$, $\frac{1}{2} \cdot 2^3 (2^3 - 1) = 28$, $\frac{1}{2} \cdot 2^5 (2^5 - 1) = 496$ as perfect numbers, and so in general for $\frac{1}{2} \cdot 2^p (2^p - 1)$ where $2^p - 1$ is prime,—a rule which holds for the first eight perfect numbers but is not universal." It is well-known that this rule is universal and that a proof thereof appears in Euclid's *Elements*. In view of the fact that perfect numbers have received so much attention in histories of elementary mathematics, it may be somewhat surprising to find on this and the following page erroneous statements relating thereto.

On page 80 there appears the following statement: "One of the most striking features of ancient arithmetic is the rarity of large numbers. There are exceptions, as in some of the Hindu traditions of Buddha's skill with numbers, in the records of some of the Babylonian tablets, and in the *Sand Reckoner* of Archimedes, with its number system extending to 10^{63}." On page 5 it is stated that Archimedes proceeded as far as 10^{52}. This disparity is, however, insignificant in comparison with the enormously larger numbers considered by Archimedes, as may be inferred from the following sentence: "Archimedes showed that his system would enable any number to be expressed up to that which in our notation would require 80,000 million million ciphers and then proceeded to prove

that this system more than sufficed to express the number of grains of sand which it would take to fill the universe, on a reasonable view (as it seemed to him) of the size to be attributed to the universe."—Heath: *A History of Greek Mathematics*, Volume 2 (1921), page 81.

On page 226 there appears a somewhat remarkable sentence as follows: "Influenced by the notion that only fractions could deal with fractions medieval writers often substituted for the division of a fraction by an integer the process of multiplying by the reciprocal of the integer; that is, $\frac{2}{3} \div 4 = \frac{2}{3} \times \frac{1}{4}$." As mathematicians have dealt with fractions since very early times, it would seem to follow from the wording of this quotation that mathematicians are fractions. Judging from the illustrative example, the quotation seems to imply that medieval writers had the notion that fractions could not be multiplied or divided by integers, but it seems unreasonable to assume that such a notion was widely adopted. The fact that instead of dividing by an integer we may multiply by its reciprocal can be explained in a much more rational way.

In view of the fact that the division of the circle into 360 equal parts has been explained in a variety of ways, it may be of interest to note here that on page 230 it is stated that "the Babylonians divided the circle into 8, 12, 120, 240 and 480 equal parts, but not into 360 such parts." This may be compared with the following: "While the sexagesimal division was used very early in measuring time the duodecimal division was retained for a long time in measuring the circle. It has not yet been determined when sexagesimal division secured also here the preference; the first definite evidence of this advance appears on clay-tablets from the time of Cambyses (about 522 B.C.). The Babylonian time-degree which appears here corresponds to four of our minutes of time so that a complete revolution, *i.e.* 24 of our hours, gives 360 such Babylonian degrees."—J. Tropfke: *Geschichte der Elementar-Mathematik*, Volume I (1921), page 39.

On pages 257 and 258 there appear various statements about negative numbers which are not in accord with the common views on this subject. For instance, on the latter page it is stated that "In India the negative number is first definitely mentioned in the works of Brahmagupta (c. 628). He speaks of 'negative and affirmative quantities' using them always as subtrahends but giving the usual rules of signs." It is

well-known that many of the best mathematical historians believe that Brahmagupta used pure negative numbers and designated such numbers by placing a dot above the corresponding natural numbers. On the preceding page it is stated that "No trace of the recognition of negative numbers, as distinct from simple subtrahends has yet been found in the writings of the ancient Egyptians, Babylonians, Hindus, Chinese, or Greeks." In view of the fact that negative numbers are so forcibly illustrated by debts, it is remarkable that they were not finally accepted as legitimate numbers until about the close of the eighteenth century, but they were used much earlier and traces of their recognition are at least as old as the works of Brahmagupta.

On page 262 it is stated that "The next to attack the problem was Bombelli (1572). In his algebra he speaks of such quantities as $+\sqrt{-a}$ and $-\sqrt{-a}$, but he made no advance upon Cardan's theory." On the contrary, Bombelli made a very decided advance as regards the theory of complex numbers. This fact has been emphasized in a recent article by H. Wieleitner, published in the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Volume 36, page 74, where the advances made by Bombelli are carefully considered and it is noted that our general histories of mathematics have failed to give due credit to Bombelli as regards the development of the fundamental concept of complex numbers.

In speaking of *La Géomètrie* by Descartes, it is stated on page 323 that "The second book defines two classes of curves, the geometric and the mechanic. We might now define the former as curves in which dy/dx is an algebraic function, and the latter as curves in which it is a transcendental function." In his review of this work published in *Il Bollettino di Mathematica*, September 1929, page lxiii, G. Loria says that he does not know where our author obtained this criterion to distinguish the geometric and the mechanic curves. When $y = \log x$, $dy/dx = 1/x$ and hence it is algebraic while the curve is transcendental. The geometric curves of Descartes are commonly regarded as algebraic, while his mechanic curves are transcendental. The quotation at the opening of the present paragraph evidently is not equivalent to this commonly accepted view. G. Loria directs attention to several other desirable modifications in the same review of Smith's *History of Mathematics* noted above.

On page 471 there appears the following statement: "The law which asserts that the equation $X = 0$, complete or incomplete, can have no more real positive roots than it has changes of signs, and no more

real negative roots than it has permanences of signs, was apparently known to Cardan; but the first satisfactory statements relating to the matter are due to Harriot (died 1621) and Descartes." This statement is preceeded by the paragraph heading "Harriot's Law of Signs," and is one of the most striking errors that appear in the volume under consideration, since attention has often been called to the fact that this law is not due to Harriot. In volume 10, page 80, of the *Bibliotheca Mathematica*, G. Eneström called attention to the fact that G. W. Leibniz seems to have been the first to make this mistake and it is on this account that the error appears in a large number of German books. There is also no evidence to support the statement that Cardan apparently knew the general law even if he knew a special case thereof.

The statements on page 473 relating to the "Fundamental Theorem" do not agree with those of other writers on this subject. For instance, the sentence "The later ones of that century (the 16th) were also aware that a quadratic equation has two roots, a cubic equation three roots, and a biquadratic equation four roots," should be modified so as to read that they were aware that some equations of these degrees have the stated number of roots. A similar modification applies to the sentence which follows the one which has just been quoted. No instance has yet been noted, in the histories of mathematics, of the solution, before the 17th century, of a quadratic equation having two negative roots, and it was not until about the middle of the eighteenth century that it was proved that the general formula for the solution of the cubic equation actually represents three roots of this equation. During the sixteenth century the imaginary roots were not as clearly understood as is implied by the given quotation.

The statement on page 514 that "Briggs introduced (1624) the word mantissa" is not in accord with any known evidence, and is of considerable interest in view of the fact that the word mantissa is so commonly used in the text-books on elementary mathematics, nor is it true that this word appears in Wallis's Algebra (1685) as is stated in the same paragraph. It does appear in the Latin edition of this algebra which was published about eight years later. What is, however, still more misleading on this page is the following quotation, which is given approvingly: "The invention of logarithms came on the world as a bolt from the blue. No previous work had led up to it, foreshadowed it or heralded its arrival. It stands isolated, breaking in upon human thought abruptly without borrowing from the work of other intellects or following known lines of mathematical thought." On the contrary, the ideas involved in

logarithms can be traced back to Euclid's *Elements* and the works of Archimedes.

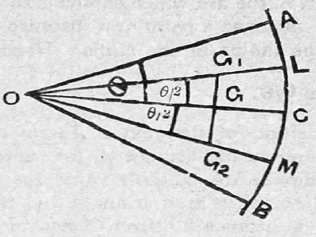
The preceding notes aim merely to throw additional light on a few points in the volume of the history of mathematics under consideration. They do not aim to be complete nor do they aim to express any opinion on the volume as a whole. Such an opinion must evidently be based upon a consideration of both the strong points and the weak points. Fortunately, the history of mathematics is growing rapidly and in view of of its enormous extent it is extremely difficult to avoid errors. It is very desirable that such errors should not be repeated and a candid discussion thereof is probably the best means to attain this end. A somewhat similar list of notes relating to volume I of the history under consideration appeared in *School Science and Mathematics*, Volume 24, (1924), page 939.

G. A. MILLER.

A Note on Centre of Gravity.

In any standard text book of Statics we can find methods for obtaining the centre of gravity of a circular arc or a sector of a circle which depend finally on the evaluation of a simple integral. But here I give a method to find the same without having recourse to integration.

Let AB be an arc of a circle whose centre is O, and radius OA, and C the midpoint of the arc AB and angle AOB = 2θ. It is obvious that the centre of gravity G of arc AB lies on OC and also the distance OG depends on the angle 2θ. Let OG be equal to some function of 2θ, say $OG = \bar{X} = f(2\theta)$. The value of $\bar{X}$ can be found out in another way. Let L and M be the midpoints of the arcs AC and CB. Then the centre of gravity of AC lies on OL and



that of CB on OM. If G_1 and G_2 are the centres of gravity of the arcs AC and CB, $OG_1 = OG_2 = f(\theta)$; because AC and CB are equal arcs of the same circle each subtending an angle θ at the centre. Now it is clear that the centre of gravity of two equal particles kept at the points G_1 and G_2 coincides with that of the arc AB. The line joining G_1 and G_2

passes through G and is also perpendicular to OC. From the figure we get

$$\bar{X} = OG = f(2\theta) = OG_1 \cos \theta/2 = f(\theta) . \cos \theta/2.$$

The equation $\quad f(2\theta) = f(\theta) . \cos \theta/2$

is true for all values of θ. From this we can easily deduce the form of the function :

$$f(2\theta) = f(\theta) . \cos (\theta/2)$$
$$= f(\theta/2) . \cos \theta/2 . \cos \theta/2^2.$$
$$= f(\theta/2^n) \cos \theta/2 \cos \theta/2^3 \cos \theta/2^3 \ldots \cos (\theta/2^{n+1}).$$

As n tends to infinity, the right-hand side expression tends to a definite limit, provided limit of $f(x)$ as x tends to 0 exists.

The left-hand side expression is independent of n. Therefcre it remains unaltered. Let limi of $f(x)$ as x tends to 0 be a and we know that the limit of $\cos \theta/2 \cos \theta/2^2 \cos \theta/2^3 \ldots \cos \theta/2^n$, as n tends to infinity is $\sin \theta/\theta$. Hence

$$f(2\theta) = \underset{n\to\infty}{\text{Lt.}} f(\theta/2^n) . \underset{n\to\infty}{\text{Lt.}} (\cos \theta/2 \cos \theta/2^2 \cos \theta/2^3 \ldots \cos \theta/2^{n+1}).$$

$$\therefore \quad f(2\theta) = a . \sin \theta/\theta.$$

In this case it is very easy to find the value of a; because $a = f(0)$ is distance from O of the centre of gravity of the arc which subtends zero angle at the centre O, That arc can be taken as a point at a distance r from O. Therefore $a = r$ where r is the radius of the circle. Hence we get the result

$$f(2\theta) = r . \sin \theta/\theta.$$

The same procedure gives the position of the centre of gravity of a sector provided we take $f(2\theta)$ to represent the distance of the centre of gravity of the sector from O and choose the suitable value for a When θ is very small, the sector can be taken as a triangle and the centre of gravity of the triangle is at a distance $\frac{2}{3} r$ from O measured along OC. Therefore $a = f(0) = \frac{2}{3} r$.

$$\therefore \quad f(2\theta) = \tfrac{2}{3} r . \sin \theta/\theta.$$

The same method can be made use of to find the centre of gravity of a surface or a solid of revolution.

Let a surface be generated by an arc AB, the axis of rotation being a line LM which does not intersect the arc and which lies in the plane of the curve. Let the angle of rotation be 2θ. The centre of gravity of the surface generated must lie on the plane containing the line LM and bisecting the angle 2θ, and its distance from LM must be some function of 2θ, say $f(2\theta)$. The method given above shows that

$$f(2\theta) = a \cdot \sin\theta/\theta.$$

where a is some constant whose value must be found out separately.

The method of finding out the centre of gravity of a solid of revolution is exactly the same as the above one with proper meanings attached to $f(2\theta)$ and a. In all such cases as I have considered now it is quite obvious that the distance of the centre of gravity from the axis of rotation varies as $\dfrac{\sin\theta}{\theta}$ where 2θ is the angle of rotation. The above method holds good even when the density is varying and is some function of the distance from the axis of rotation.

In the last two cases there is some difficulty in finding out the value of the constant a. If we know the position of the centre of gravity for some one value of θ, a is easily calculated. For example, let us take a semi-circle and rotate it through an angle 2θ about its bounding diameter. Distance from the axis of the centre of gravity of the surface generated

is $\quad a \cdot \dfrac{\sin\theta}{\theta} = f(2\theta)$. When $\theta = \dfrac{\pi}{2}$ we know that $f(2\theta) = \dfrac{r}{2}$.

Therefore $\quad \dfrac{r}{2} = a\,\dfrac{2}{\pi}$, *i.e.* $a = \dfrac{\pi r}{4}$. $\therefore\ f(2\theta) = \dfrac{\pi r}{4} \cdot \dfrac{\sin\theta}{\theta}$.

The distance from the axis of the centre of gravity of the solid generated is given by $f(2\theta) = a \cdot \dfrac{\sin\theta}{\theta}$; but $f(2\theta) = \dfrac{3r}{8}$ when $\theta = \dfrac{\pi}{2}$.

$$\therefore\ \frac{3r}{8} = a \cdot \frac{2}{\pi}, \qquad i.e. \quad a = \frac{3\pi r}{16}.$$

This gives the formula $f(2\theta) = \dfrac{3\pi r}{16} \cdot \dfrac{\sin\theta}{\theta}$

Finding the Centre of Gravity by this method requires the knowledge of its position in a particular case or for some one value of θ, say $\theta = \pi/2$.

Mysore University S. P. RANGANATHACHAR.

Solutions.

Question 699.

(S. RAMANUJAN):—Show that the roots of the equations

(i) $x^6 - x^3 + x^2 + 2x - 1 = 0,$

(ii) $x^6 + x^5 - x^3 - x^2 - x + 1 = 0$

can be expressed in terms of radicals.

Solution by Prof. G. N. Watson.

It seems unlikely that Ramanujan inserted the trivial factors $x + 1$ and $x - 1$ in order to give the impression to anyone who noticed them that he had made a substantial advance towards solving the equations.

The essential equations are

(1) $$x^5 - x^4 + x^3 - 2x^2 + 3x - 1 = 0$$

and

(2) $$x^5 + 2x^4 + 2x^3 + x^2 - 1 = 0.$$

These are both equations which occur in connexion with 'singular moduli' in the theory of complex multiplication of elliptic functions.

If k is the modulus for which the periods are connected by the relation

$$K' = K \sqrt{79},$$

then $\sqrt[12]{(16kk')}$ is the real root of (1): and (2) is the corresponding equation when $\sqrt{79}$ is replaced by $\sqrt{47}$. The former equation has been constructed by Russell and by Greenhill, the latter by Greenhill; but in constructing (2), Greenhill has made an obvious mistake in sign, his equation being the equation which is satisfied by $-\sqrt[12]{(16kk')}$; the equation is given correctly by Weber.*

It was remarked by Abel (see his *Oeuvres Complètes*, **1**, p, 272) that equations of this type satisfied by singular moduli are soluble by radicals; and this is the proof of Ramanujan's two theorems.

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* Russell : *Proc. London Math. Soc.* (1) 21 (1891), p. 359, Greenhill : *ibid.* pp. 407, 408. Weber : *Lehrbuch der Algebra*, 3 (1908), p. 723.

At the risk of being considered irrelevant, I shall make some further remarks about these equations. Both (1) and (2) are to be found in Ramanujan's unpublished note-books: but while he gives the modular equation of order 47, from which (2) is easily derived, there is no mention of the modular equation of order 79 apart from equation (1); if I am correct in my surmise that I have reconstructed Ramanujan's work on modular equations more or less correctly, the equation of order 47 appears as part of his general theory.

It is also irrelevant, but it may be of some interest to give the actual solutions of (1) and (2), which are somewhat complicated; the solution of (2) was completely worked out by G. P. Young,* after it was "brought under the notice of the writer by a mathematical correspondent" (possibly Greenhill); the solution was simplified by Cayley † whose result is that the real root of (2) is

$$\sqrt{\frac{\sqrt[5]{A} + \sqrt[5]{B} + \sqrt[5]{C} + \sqrt[5]{D}}{10}}$$

where

$$A, D = 39000 + 18200\sqrt{5} \pm (1720 + 920\sqrt{5})\sqrt{(235 + 94\sqrt{5})},$$

$$B, C = 39000 - 18200\sqrt{5} \mp (1720 - 920\sqrt{5})\sqrt{(235 - 94\sqrt{5})}.$$

To obtain the corresponding solution of (1), put $x^2 = 1/\xi$ and we get

$$\xi^5 - 5\xi^4 - 3\xi^2 - \xi - 1 = 0, \tag{3}$$

and if we next put $\xi = X + 1$, we get

$$X^5 - 10X^3 - 23X^2 - 22X - 9 = 0. \tag{4}$$

This equation can be solved by the method developed by Young; the success of the method depends on detecting simultaneous rational solutions of two subsidiary equations, which in the case of (4) are

$$50y^2t + y\left(8t^3 + \frac{46}{5}t^2 - \frac{56}{5}t + 9\right) + \frac{11}{50}t - \frac{927}{1000} = 0.$$

$$25y^3 - y^2\left(16t^4 + 56t^2 - \frac{368}{5}t + \frac{43}{5}\right) - y\left(\frac{22}{25}t^2 - \frac{109}{50}\right) - \frac{121}{10000} = 0.$$

* *American J. of Math.*, 10 (1888), pp 99—130.

† *American J. of Math.*, 13 (1891), pp. 53—58.

No solutions of these being very obvious, and the task of eliminating t from these equations being formidable, I noticed that, if ϕ is a root of Cayley's* sextic resolvent, then $500y = \phi^2$, and this makes it unnecessary to perform the elimination (the result of the elimination is given by Young only in the case when the co-efficient of X^3, as well as the co-efficient of X^4, in the quintic is zero).

It is less troublesome to compute the resolvent by using (3) rather than (4), and the result is

$$(5) \quad \phi^6 + 140\,\phi^4 - 3680\,\phi^2 + 22752\,\phi - 72064 = 0;$$

the discriminant of the quintic, which is involved in the co-efficient of ϕ, is equal to $711^2/5^5$, and this contains 79 as a repeated factor. The rational root of (5) is $\phi = 4$, by inspection, and hence the solutions of the subsidiary equations are

$$y = \frac{4}{125},\ t = \frac{9}{8}.$$

The remainder of the work, following Young's method, is quite straightforward, though the arithmetic is rather heavy; it seems sufficient to state the result, namely that the solution of (1) is given by

$$\frac{1}{x^2} - 1 = X = \sqrt[5]{A} + \sqrt[5]{B} + \sqrt[5]{C} + \sqrt[5]{D},$$

where

$$\begin{matrix} 2500\,A \\ 2500\,D \end{matrix} = 6525 + 2235\sqrt{5} \pm (2 + \sqrt{5})^4 . \sqrt{(790 + 158\sqrt{5})}$$

$$\begin{matrix} 2500\,B \\ 2500\,C \end{matrix} = 6525 - 2235\sqrt{5} \mp (2 - \sqrt{5})^4 . \sqrt{(790 - 158\sqrt{5})}$$

As a numerical check we have

$$\begin{aligned} A &= 8\cdot9640786 & \sqrt[5]{A} &= 1\cdot5506003 \\ B &= 0\cdot6109293 & \sqrt[5]{B} &= 0\cdot9061460 \\ C &= 0\cdot6109812 & \sqrt[5]{C} &= 0\cdot9061601 \\ D &= 0\cdot2540109 & \sqrt[5]{D} &= 0\cdot7602746 \end{aligned}$$

$$X = 4\cdot1231811$$

$$\begin{aligned} 22X &= 90\cdot710 & X^5 &= 1191\cdot687 \\ 23X^2 &= 391\cdot014 \\ 10X^3 &= 700\cdot966 \\ \hline & 1182.690 \end{aligned}$$

and this verification showing agreement to six significant figures seems to afford an adequate check of the correctness of my work.

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* *Phil. Trans. Royal Soc.* **151 (1861)**, pp **263—276.**

Question 1444.

(SELECTED) :—Prove that

$$\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\ldots=\frac{\Theta^2}{1^3}+\frac{\Theta^4}{2^3}+\frac{\Theta^6}{3^3}+\ldots\ldots$$

where $$\Theta = 2\sin\frac{\pi}{10}.$$

Solution by Prof. G. N. Watson.

As has been pointed out by Mr. S. D. Chowla, this result is incorrect.

If $$\phi(x)=\frac{x}{1^3}+\frac{x^2}{2^3}+\frac{x^3}{3^3}+\ldots\ldots,$$

the correct result is

$$\phi(\Theta^2)=\frac{4}{5}\phi(1)+\frac{2\pi^2}{15}\log\left(2\sin\frac{\pi}{10}\right)-\frac{2}{3}\left\{\log\left(2\sin\frac{\pi}{10}\right)\right\}^3$$

not $\phi(\theta^2)=\phi(1)$, as is asserted.

To establish the result we need two formulae ; the first,

(1) $$\phi(x)+\phi(-x)=\tfrac{1}{4}\phi(x^2),$$

is obvious ; the second,

(2) $$\phi\left(\frac{x}{x-1}\right)+\phi(x)+\phi(1-x)=\phi(1)+\frac{\pi^2}{6}\log(1-x)+\frac{1}{6}\left\{\log(1-x)\right\}^3-\frac{1}{2}\log x\left\{\log(1-x)\right\}^2.$$

was proved by Spence; and I have published another proof in the *Journal, London Math. Soc.*, 3 (1928), pp. 217-218.

Observe that $\theta^2=1-\theta$ when $\Theta=2\sin\frac{\pi}{10}$ and put θ^2 for x in (2);

since $$\frac{\Theta^2}{\Theta^2-1}=-\theta,$$

formula (2) becomes

$$\phi(-\theta)+\phi(\theta^2)+\phi(\theta)=\phi(1)+\frac{\pi^2}{6}\log\theta+\frac{1}{6}(\log\theta)^3-\frac{1}{2}\log\theta^2.(\log\theta)^2,$$

and hence from (1)

$$\frac{5}{4}\phi(\Theta^2)=\phi(1)+\frac{\pi^2}{6}\log\theta-\frac{5}{6}(\log\theta)^3,$$

which is equivalent to the corrected result.

I have checked this arithmetically from the following values which

are correct to seven places of decimals :

$$\phi(\theta^2) = 0{\cdot}4026840, \qquad \frac{4}{5}\phi(1) = 0{\cdot}9616455,$$

$$-\frac{2\pi^2}{15}\log\theta = 0{\cdot}6332494, \quad -\frac{2}{3}(\log\theta)^3 = 0{\cdot}0742878.$$

Question 1466.

(G. A. SRINIVASAN) :—Required a proof by purely elementary methods (without the use of multiplication of series) that

$$E(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \dots$$

is positive for all real values of x.

Solution by B. B. Bagi.

It is necessary to prove the result only for negative values of x. Denote the sum of the first $(2n+1)$ terms of the series by $S_{2n}(x)$ and the sum of the remaining terms by $R_{2n}(x)$. It is then easy to prove by Van der Monde's theorem that

$$S_{2n}(x) \times S_{2n}(-x) = 1 + 2x^{2n+2}\left\{\frac{1}{(1)!\,(2n+1)!} - \frac{1}{(2n+2)!}\right\}$$

$$+ 2x^{2n+4}\left\{\frac{1}{(3)!\,(2n+1)!} - \frac{1}{(2)!\,(2n+2)!}\right.$$

$$\left. + \frac{1}{(1)!\,(2n+3)!} - \frac{1}{(2n+4)!}\right\} + \dots \dots$$

Now we observe that

$${}^{2r}C_0 < {}^{2r}C_1 < {}^{2r}C_2 < \dots \dots < {}^{2r}C_{r-1}.$$

Hence $S_{2n}(x) \times S_{2n}(-x)$ is an even polynomial with positive co-efficients. It is therefore positive for all real values of x. It follows that $S_{2n}(x)$ is positive for all real values of x.

Now $S_{2n}(-x)\,S_{2n}(x) = 1 +$ a polynomial of even powers of x.

Hence $$S_{2n}(-x)\,S_{2n}(x) > 1.$$

But $S_{2n}(x)$ is bounded above, x being any positive number ; in fact

$$S_{2n}(x) < E(x).$$

Hence $$S_{2n}(-x) > \frac{1}{S_{2n}(x)} > \frac{1}{E(x)}.$$

Hence, as n tends to infinity, $S_{2n}(-x)$ tends to a positive limit. Also Lim. $S_{2n+1}(-x) =$ Lim. $S_{2n}(-x) = E(-x)$.

Question 1468.

(S. D. CHOWLA):—Let $N = p^{\alpha}q^{\beta}r^{\gamma}\ldots\ldots$, $\chi(N) = \alpha\beta\gamma\ldots\ldots$, $[x]$ = integer *nearest* to x, then prove that

$$\sum_{N/d^6} \chi\left(\frac{N}{d^6}\right) = \left[\frac{(\alpha+3)^2}{12}\right]\left[\frac{(\beta+3)^2}{12}\right]\left[\frac{(\gamma+3)^2}{12}\right]\ldots\ldots$$

the summation extending to all the divisors d^6 of N which are perfect sixth powers.

Solution by K. V. Vedantham.

$[x]$ should be taken as the integral part of x and not the integer nearest to x.

Let $\alpha = n\rho_1 + d_1$, $\beta = n\rho_2 + d_2$, $\gamma = n\rho_3 + d_3 \ldots$ where all are integers and $d_1, d_2, d_3 \ldots$ are all less than n. The different values of α in the summation are d_1, $n + d_1$, $2n + d_1$, ... $\rho_1 n + d_1$; of β, d_2, $n + d_2$, $2n + d_2$, ... $\rho_2 n + d_2$, and so on. Therefore the summation is $(d_1 + n + d_1 + \ldots + n\rho_1 + d_1)(d_2 + n + d_2 + \ldots + n\rho_2 + d_2)\ldots$

$$= \left(\rho_1 + 1\right)\left(\frac{n\rho_1}{2} + d_1\right)\left(\rho_2 + 1\right)\left(\frac{n\rho_2}{2} + d_2\right)\ldots\ldots$$

$$= \left(\frac{(2\alpha+n)^2 - (2d_1 - n)^2}{8n}\right)\left(\frac{(2\beta+n)^2 - (2d_2-n)^2}{8n}\right)\ldots\ldots$$

Since $d_1, d_2, d_3 \ldots$ are all less than n,

$$\frac{(2d_1 - n)^2}{8n}, \frac{(2d_2-n)^2}{8n}\ldots\ldots$$

are all less than $8n$. When $n < 8$,

$$\frac{(2d_1-n)^2}{8n}, \frac{(2d_2-n)^2}{8n}, \text{ are all fractions.}$$

$$\therefore \frac{(2\alpha+n)^2 - (2d_1-n)^2}{8n} = \left[\frac{(2\alpha+n)^2}{8n}\right]$$

where $[x]$ is the integral part of x and similarly for other factors.

When $n = 6$, this becomes $\left[\frac{(2\alpha+6)^2}{48}\right] = \left[\frac{(\alpha+3)^2}{12}\right]$.

Hence in the present case

$$\sum_{N/d^6} \chi\left(\frac{N}{d^6}\right) = \left[\frac{(\alpha+3)^2}{12}\right]\left[\frac{(\beta+3)^2}{12}\right]\left[\frac{(\gamma+3)^2}{12}\right]\ldots\ldots$$

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1573. (V. RAMASWAMI AIYAR, M.A.):—Show that the series

$$1 + \frac{z(z+1)^0}{1!\ e} + \frac{z(z+2)^1}{2!\ e^2} + \frac{z(z+3)^2}{3!\ e^3} + \dots\dots$$

is absolutely convergent for all values of z and that its value is e^z.

1574. (V. RAMASWAMI AYAR, M.A.):—In a given triangle ABC, AX is drawn perpendicular to BC and two points P, Q are taken thereon such that P, Q, B, C are mutually orthocentric. Show that the pedal circles of P, Q with respect to ABC touch one another at X. Also if T be the point in BC whose pedal circle with respect to ABC touches those of P and Q at X, then TP, TQ envelope a hyperbola of which XA, XA′ are asymptotes and AA′ is a tangent, where AA′ is the diameter through A of the circumcircle of ABC.

1575. (B. RAMAMURTHI, B.A. (Hons.)):—ABC, A′B′C′ are two cubicovariant triads on a twisted cubic. H_1, H_2 are the Hessian points. The tangent at H_1 meets the osculating plane at H_2 in P and the tangent at H_2 meets the osculating plane at H_1 in Q. Show that the planes ABC, A′B′C′ and the osculating planes at H_1 and H_2 form a harmonic axial pencil. Show also that the quadrics through ABC, A′B′C′ and P, Q touch at P and Q, the tangent planes being PH_1H_2 and QH_1H_2.

1576. (A. RANGANATHA RAO):—O is any point on the nine points circle of a triangle ABC and D, E, F are the middle points of BC, CA, AB respectively. Prove that the lines through the orthocentre of ABC, parallel to OD, OE, OF meet BC, CA, AB respectively in three points which are collinear.

1577. (S. AUDINARAYANAN):—If Ω is the polar cubic of a point P with respect to a tetrahedron ABCD, show that the polar quadric of P with respect to ABCD is the locus of points Q which are such that on the polar quadric of Q with respect to Ω, ABCD are such that the tangent plane at A passes through the Hessian points of BCD.

UNSOLVED QUESTIONS FOR WHICH SOLUTIONS ARE INVITED.

266. (N. DURAIRAJAN) :—Describe a rectangular hyperbola so as to touch four concentric circles.

267. (N. DURAIRAJAN) :—A straight line cuts a plane cubic curve in three points P, Q, R and the mean centre of P, Q, R is on the cubic. Find the envelope of the line PQR.

327. (S. RAMANUJAN) :—Show that the Euler's constant, *viz.*

$$1 + \tfrac{1}{2} + \tfrac{1}{3} \dots + \frac{1}{n} - \log n$$

when n is infinite, is equal to

$$\log 2 - 1\left(\frac{2}{3^3 - 3}\right) - 2\left(\frac{2}{6^3 - 6} + \frac{2}{9^3 - 9} + \frac{2}{12^3 - 12}\right)$$

$$- 3\left(\frac{2}{15^3 - 15} + \frac{2}{18^3 - 18} + \dots + \frac{2}{39^3 - 39}\right) \dots \dots$$

the first term in the nth group being

$$\frac{2}{\left(\frac{3^n + 3}{2}\right)^3 - \left(\frac{3^n + 3}{2}\right)}$$

350. (K. J. SANJANA M.A.) :—Sketch the curve given by

$$x = \phi \sin \alpha - \cos \phi, \quad y = \cos \alpha + \sin \phi$$

where ϕ is the parameter and find its inflexions.

Obtain the inflexions of

$$25x^3 = 25y^2 - 290xy - 504x^2 - 7000y.$$

352. (S. RAMANUJAN) :—Show that

$$\text{(i)} \quad \frac{1}{1-}\,\frac{e^{-2\pi}}{1+}\,\frac{e^{-4\pi}}{1+}\,\frac{e^{-6\pi}}{1+}\,\dots = \left\{\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right\}\sqrt[5]{e^{2\pi}}$$

$$\text{(ii)} \quad \frac{1}{1-}\,\frac{e^{-\pi}}{1+}\,\frac{e^{-2\pi}}{1-}\,\frac{e^{-3\pi}}{1+}\,\dots = \left\{\sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}\right\}\sqrt[5]{e^{\pi}}.$$

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LIST OF JOURNALS & BOOKS RECEIVED IN THE LIBRARY

during the months of August and September 1930.

1 Abhandlungen aus dem Seminar, Hamburg, **8**, 1.
2 American Journal of Mathematics, **52**, 3.
3 American Mathematical Monthly, **37**, 3, 6.
4 Annals of Mathematics, **31**, 3.
5 Astrophysical Journal, **71**, 3, 4, 5.
6 Bulletin of the American Mathematical Society, **36**, 6. 7.
7 Bulletin of the Calcutta Mathematical Society, **22**, 1.
8 Bulletin des Sciences Mathematiques, **54**, June.
9 Crelle's Journal, **162**, 3, 4.
10 Japanese Journal of Mathematics. **7**, 1.
11 Mathematical Gazette, **15**, 208 (3 copies)
12 Mathematische Annalen, **103**, 3.
13 Monthly Notices of the Royal Astronomical Society, **90**, 8.
14 Philosophical Magazine, **10**, 62, 63, 64
15 Philosophical Transactions of the Royal Society, **229**, pp. 255—328.
16 Popular Astronomy, **38**, 7.
17 Proceedings of the Edinburgh Mathematical Society, **2**, 2.
18 Proceedings of the London Mathematical Society, **31**, 2, (3 copies). **31** (2 copies)
19 Proceedings of the Physico-Mathematical Society of Japan, **12**, 6
20 Proceedings of the Royal Society of London, **128**, 807 (2 copies) **128** (2 copies).
21 Quarterly Journal of Mathematics, **1**, 2.
22 Rendiconti del Circolo Mathematico di Palermo, **54**, 1.
23 Tohoka Mathematical Journal, **32**, 3, 4.
24 Transactions of the Royal Society of South Africa, **18**, 4.

Books.

1 The Zeta Functions of Riemann, by E. C. Titchmarsh M A
2 Teoria Geometrica de La Polaridad, by D. Julis Rey Pastor

Publication de La Faculte des Sciences.

1 Synthesis of cyclopentyl substituted aliphatic acid studies in resolution and an X-Ray investigation of enantiomorphism, by Gail Robert Yohe.
2 University Tutorial Press Catalogue of Educational Publications
3 Reprints from the Transactions of the Royal Society of South Africa, Vol. 18, Parts 2, 3.
4 Mathematical Notes published by the Edinburgh Mathematical Society.
5 Growth of Mathematics on the Continent during the 18th century, By Rev. H. Rafael.

Numbers in black type refer to the volumes, and those in ordinary type to the numbers of the issues

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